Construction of the $\eta \to 3\pi$ (and $K \to 3\pi$) amplitudes using dispersive approach

M. Zdráhal[ab], K. Kampf[ac], M. Knecht[d] and J. Novotný[a]

[a] IPNP, Charles University, Czech Republic
[b] Faculty of Physics, University of Vienna, Austria
[c] PSI, Switzerland
[d] CPT, CNRS Marseille, France

ABSTRACT

The $\eta \to 3\pi$ and $K \to 3\pi$ decays are important processes for studies of the isospin breaking effects. Nowadays, the so-called cusp effect in the K decays is analyzed at KTeV and NA48 experiments. From the theoretical point of view this was studied recently by different methods. We propose a unified and relativistic treatment relying on very general principles, unitarity, analyticity and crossing symmetry, combined with chiral counting, in order to construct model-independent representations of the corresponding amplitudes that are valid at two loops.

1 Motivation

During the last few years, the decay processes $K \to 3\pi$ and $\eta \to 3\pi$ have been under intensive studies, both from the experimental and from the theoretical points of view. For the sake of limited space here, we refer the reader for a more comprehensive list of these works to our proceedings [1]. Nevertheless, we list the experimental collaborations undertaking these measurements [2] and theoretical studies dealing with $\eta \to 3\pi$ decay [3]–[4] as we concentrate here mainly on results of this decay.
The importance of these processes, besides the usual determination of decay rates and energy distributions, lies in the possibility of studying isospin breaking effects. The appearance of the cusp effect in those processes with two neutral pions in the final state enables quite a simple determination of the $\pi\pi$ scattering lengths (mainly from $K^+ \to \pi^+\pi^0\pi^0$ decay). The η decays, which are forbidden in the isospin limit, offer a good possibility to determine the isospin breaking parameters like $R = \frac{m_s - \hat{m}}{m_d - m_u}$.
The processes in question are

$$K^+ \to \pi^+\pi^0\pi^0,\ \pi^+\pi^+\pi^-; \qquad K_L \to \pi^0\pi^0\pi^0,\ \pi^0\pi^+\pi^-; \qquad \eta \to \pi^0\pi^0\pi^0,\ \pi^0\pi^+\pi^-; \qquad K_S \to \pi^0\pi^+\pi^-. \tag{1}$$

The standardly used model-independent parametrization of the energetic dependence of the Dalitz plot for these decays (Dalitz parametrization of PDG) neglects the unitarity corrections coming from the final state interactions.

2 Our parametrization

Up to two loops our parametrization takes the form

$$\mathcal{A}(s,t,u) = \mathcal{N}_F\left[\mathcal{P}(s,t,u) + \mathcal{U}(s,t,u)\right] + O(p^8), \tag{2}$$

where $\mathcal{N}_F$ is an overall normalization, while the polynomial part $\mathcal{P}(s,t,u)$ contains free parameters describing the energy dependence of the processes in analogy to the Dalitz parameters. All the non-analytic part of the amplitude connected with the final state interactions is contained in $\mathcal{U}(s,t,u)$. It depends on the parameters from the polynomial part and on parameters describing the intermediate interactions. In the low-energy region and up to two loops, the only such interactions one has to take into account are the $\pi\pi$ rescatterings – the intermediate states other than the ones with two pseudo-Goldstone bosons are suppressed to the $O(p^8)$ order up to a polynomial contribution included already in $\mathcal{P}(s,t,u)$. The intermediate states like e.g. $P\pi$ can be expanded in powers of the Mandelstam variables and reasonably below the $P\pi$ threshold (where we are interested in) they can be approximated just by a polynomial contribution again included in $\mathcal{P}(s,t,u)$.

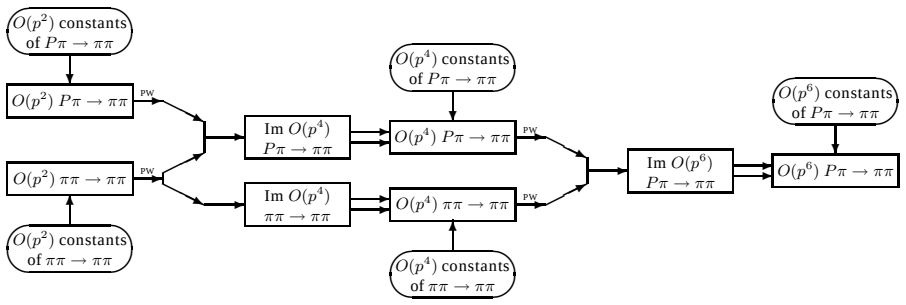
3 Reconstruction procedure

The procedure uses the methods of reconstruction theorem from [5], based on the analyticity, unitarity, crossing symmetry and chiral power-counting for partial wave amplitudes

$$\mathcal{A}(s,t,u) = 16\pi\mathcal{N}_F(f_0(s) + 3f_1(s)\cos\theta) + \mathcal{A}_{\ell\geq 2}, \tag{3}$$

$$\mathrm{Re}\, f_{\ell=0,1}(s) \sim O(p^2), \quad \mathrm{Im}\, f_{\ell=0,1}(s) \sim O(p^4), \qquad \mathrm{Re}\,\mathcal{A}_{\ell\geq 2} \sim O(p^4), \quad \mathrm{Im}\,\mathcal{A}_{\ell\geq 2} \sim O(p^8), \tag{4}$$

The theorem enables us to write the amplitude in the form (2), with $\mathcal{P}(s,t,u)$ being a third order polynomial in the Mandelstam variables having the same s,t,u symmetries as the amplitude $\mathcal{A}(s,t,u)$ and the unitarity part $\mathcal{U}(s,t,u)$ given in terms of single variable dispersive integrals over the imaginary parts of S and P partial waves of all the crossed amplitudes.
These imaginary parts are obtained from the unitarity relation projected on the corresponding partial waves. The important is that to get them to the order we are interested in, it is enough to take into account only $\pi\pi$ scattering intermediate interactions. Thus, we iteratively construct in parallel the $\pi\pi$ scattering amplitude and the amplitude of $P\pi \to \pi\pi$, related to the $P \to 3\pi$ by crossing symmetry.
A simple diagram of this construction is



We see that the final amplitude depends on the $\pi\pi$ scattering parameters, which can be also fitted from the studied process or we can simplify the fit using the values of $\pi\pi$ parameters from elsewhere.
One can also see that a given polynomial parameter has further contributions at every order and that actually the imaginary part of amplitude depends on parameters of the lower order. We can deal with this fact in two ways:

- fit order-by-order – we can exactly respect the chiral orders and distinguish contributions of every order. We can further simplify the fit by starting with fitting the lower order formula and afterwards fit only contributions of the higher orders to the polynomial part
- resummed fit – we can include also part of the neglected orders by replacing all the parameters by their highest order values. This possibility is natural in the cases where we want to give or keep the physical meaning of the parameters (this will be of a use e.g. when fitting scattering lengths from the cusp data). By that the unitarity of the result is satisfied better also nonperturbatively.

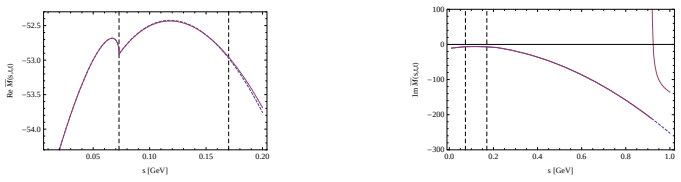
4 Results

We are finishing the $O(p^6)$ parametrizations of all the decay amplitudes (1) and preparing for the fit of experimental data.
As an illustration we are presenting here the fit to the results of Bijnens and Ghorbani [3] for $\eta \to 3\pi$ decay, calculated in the first order in isospin breaking, where the interesting physics appears already at this order.
For simplicity we do not fit the $\pi\pi$ scattering parameters, but take them from Knecht et al [5].

4.1 Order-by-order fit

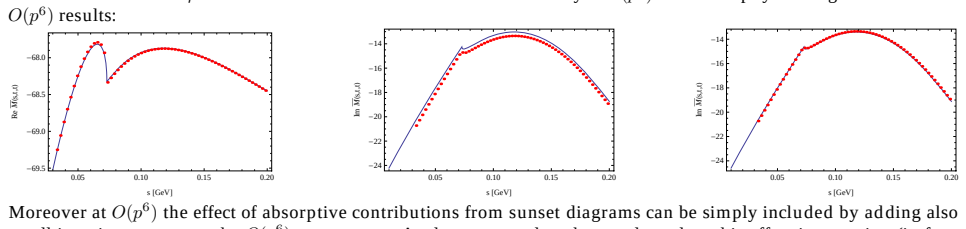
Our parametrization with parameters obtained by order-by-order fit (blue) should reproduce the chiral perturbation theory results [3] (red) on the physical decay region (indicated by the dashed lines). Here we plot one particular cut in kinematical variables ($t = u$) of the neutral case $\eta \to 3\pi^0$. Nevertheless, the figures are similar also for other cuts.

- $O(p^4)$ results:



The $O(p^4)$ imaginary part under the $\pi\eta$ and KK threshold is reproduced exactly in the case when our amplitude is defined consistently to the chiral one (GMO relations, $\pi\pi$ scattering parameters, ...).
The influence of the $\pi\eta$ and KK intermediate states can be seen already at $O(p^4)$ above the physical region.

- $O(p^6)$ results:



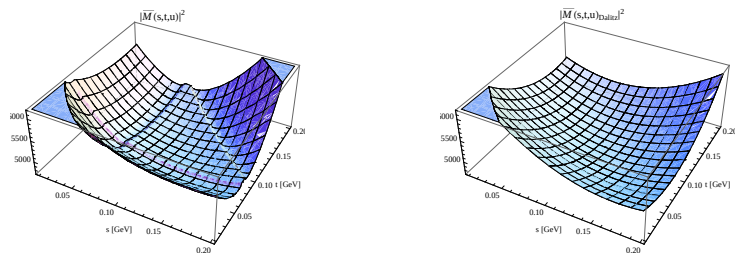
Moreover at $O(p^6)$ the effect of absorptive contributions from sunset diagrams can be simply included by adding also small imaginary parts to the $O(p^6)$ parameters. As demonstrated at the two last plots this effect is very tiny (in fact, only a constant shift is needed and is suppressed by two orders of magnitude with respect to the real part of the $O(p^6)$ contribution to the constant term – the last figure).

4.2 Order-by-order fit vs. resummed fit

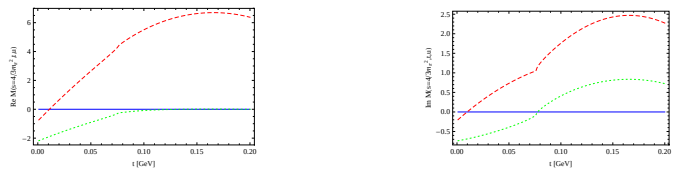
The further illustration can shed light on the sign discrepancy between χPT and dispersive determination of the Dalitz parameter α of the neutral $\eta \to 3\pi^0$ decay. The Dalitz parametrization of this process is defined according to

$$|\bar{M}(s,t,u)|^2 = \bar{A}_0^2\,(1 + 2\alpha z + \dots), \qquad z = \frac{2}{3}\sum_{i=1}^{3}\left(\frac{3E_i - M_\eta}{M_\eta - 3M_\pi}\right)^2, \tag{5}$$

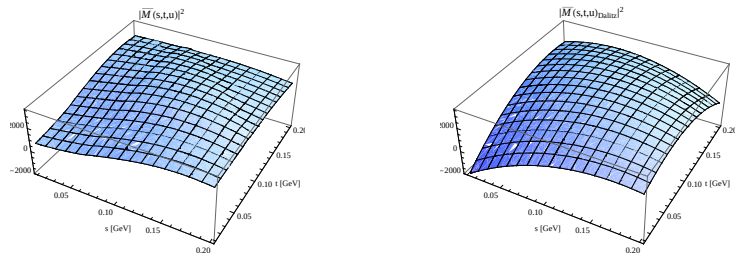
where E_i is the energy of the $i-$th pion in the final state. The χPT analysis [3] gives a positive value for α whereas the dispersive approaches from [4] (and also experiments) predict negative values.
Using the fit of χPT data from the previous subsection, we can plot Dalitz plot of $|\bar{M}(s,t,u)|^2$ dependence on s and t coming from the NNLO χPT (left) and from that determine the Dalitz parameterization (right), giving indeed a positive value of α.



To reproduce the dispersive methods, we fit our resummed fit parametrization $\eta \to \pi^0\pi^+\pi^-$ of $|M(s,t,u)|^2$ to the chiral $O(p^6)$ results at the Adler zero ($s = \frac{4}{3}M_\pi^2$), where one believes the chiral expansion. The LO(solid), NLO(dashed) and NNLO(dotted) of the real and imaginary part look like

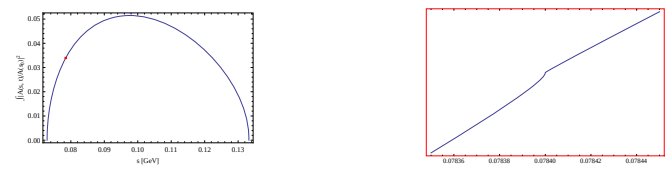


Using these resumed fit parameters, we plot the neutral Dalitz plot and from that determined Dalitz parametrization. We see that the behaviour has really changed the sign. This can also affect the computation of the decay widths of these amplitudes and change the value of R obtained.



4.3 $O(p^4)$ cusp

If we go beyond the isospin conservation limit, we can describe also the cusp effect in the decay processes with two neutral pions in the final state. Our $O(p^4)$ $K_L \to 3\pi^0$ $O(p^4)$ parametrization with a particular choice of the parameters giving a similar picture as the chiral perturbation theory result [6] gives (partial decay rate $K_L \to 3\pi^0$ (in arbitrary units) as a function of the invariant mass of the $\pi^0\pi^0$ pair squared; on the right the cusp region is zoomed.)



5 Summary

We have a general method allowing us to construct a model-independent two-loop parametrization of the amplitudes of interesting decay modes $K \to 3\pi$ and $\eta \to 3\pi$, based only on analyticity, unitarity, crossing symmetry, relativistic invariance and chiral power-counting for partial wave amplitudes.
We have verified that this description is fully compatible with the previous existing chiral two-loop calculation for $\eta \to 3\pi$ in the first order in isospin limit. It is however more general and can help to understand also the discrepancy in the sign of slope in Dalitz parametrization between standard chiral calculation and previous dispersive approaches.
Our method can be simply used also beyond the isospin limit (i.e. describing cusp, ...).

Acknowledgement : We are grateful to J. Bijnens for providing us with the Fortran programs that generate χPT $\eta \to 3\pi$ results [3].

References

[1] M. Zdrahal, K. Kampf, M. Knecht and J. Novotny, PoS **(EFT09)** 063 [arXiv:0905.4868 [hep-ph]]; K. Kampf, M. Knecht, J. Novotny and M. Zdrahal, Nucl. Phys. Proc. Suppl. **186** (2009) 334.

[2] Crystal Ball / MAMI, Crystal Barrel, KLOE, KTeV, NA48, SND, WASA / COSY, ...

[3] J. Bijnens and K. Ghorbani, J. High Energy Phys. **0711** (2007) 030.

[4] J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 539; J. Kambor, C. Wiesendanger and D. Wyler, Nucl. Phys. B **465** (1996) 215; A. V. Anisovich and H. Leutwyler, Phys. Lett. B **375** (1996) 335; B. Borasoy and R. Nissler, Eur. Phys. J. A **26** (2005) 383; M. Bissegger *et al.*, Phys. Lett. B **659** (2008) 576.

[5] J. Stern, H. Sazdjian, N. H. Fuchs, Phys. Rev. D **47** (1993) 3814; M. Knecht, B. Moussallam, J. Stern and N. H. Fuchs, Nucl. Phys. B **457** (1995) 513; M. Zdrahal and J. Novotny, Phys. Rev. D **78**, 116016 (2008).

[6] J. Bijnens and F. Borg, Nucl. Phys. B **697** (2004) 319.