Hadronic light-by-light scattering in the muon $g-2$: a new short-distance constraint on pion exchange

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Based on:

Nyffeler, Phys. Rev. D 79, 073012 (2009), arXiv:0901.1172 [hep-ph]

Jegerlehner + Nyffeler, Phys. Rept. 499, 1 (2009), arXiv:0902.3360 [hep-ph]

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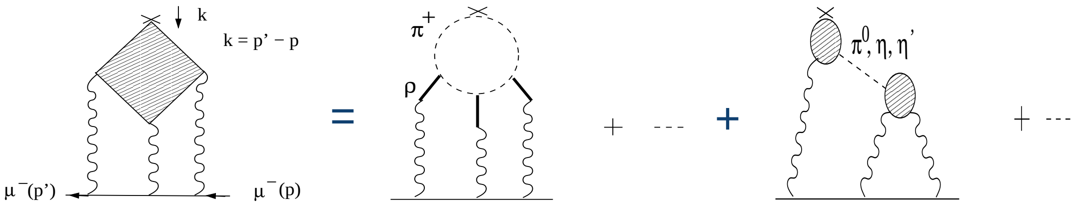
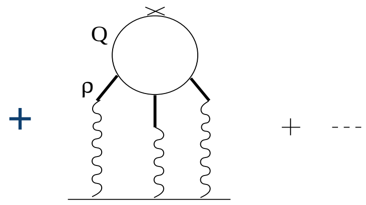
Hadronic light-by-light scattering in the muon $g-2$

Classification of contributions (de Rafael '94):



Exchange of other resonances $(f_0, a_1, \ldots)$



Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

Relevant scales ~ 200 MeV $-\ 2$ GeV. No direct relation to experimental data, in contrast to hadronic vacuum polarization in $g-2 \longrightarrow$ need hadronic (resonance) model

Contribution to $a_\mu \times 10^{11}$:

HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) $[a_1]$	+10 (11)
BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) $[f_0, a_1]$	+21 (3)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) $[a_1]$	0
2007:	+110 (40)				
PdRV:	+105 (26)	-19 (19)	+114 (13)	+8 (12) $[f_0, a_1]$	2.3 [c-quark]
		ud.: -45	ud.: $+\infty$		ud.: +60

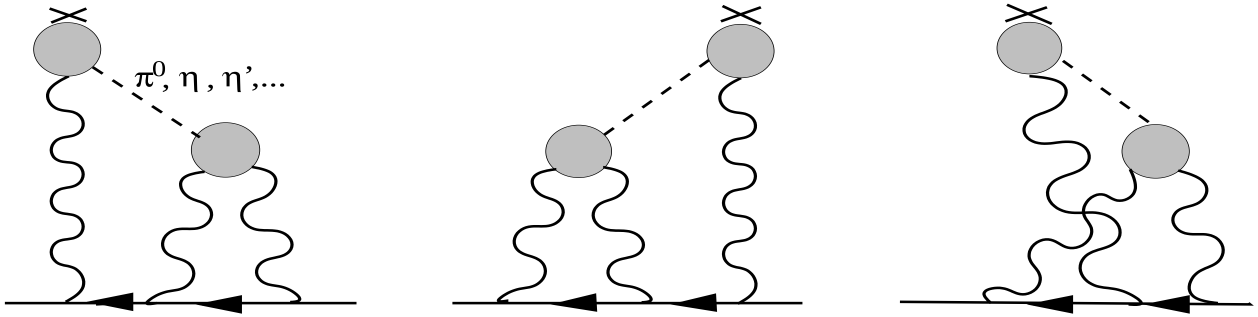
ud. = undressed, i.e. point vertices without form factors

HKS = Hayakawa, Kinoshita, Sanda; BPP = Bijnens, Pallante, Prades; KN = Knecht, Nyffeler; MV = Melnikov, Vainshtein;

2007 = Bijnens, Prades; Miller, de Rafael, Roberts

PdRV = Prades, de Rafael, Vainshtein '09: New combination of existing results. No dressed light quark loops ! Assume them to be taken into account by using short-distance constraint of M+V '03 on pseudoscalar-pole contribution. Why should this be the case ?

Pseudoscalar-exchange contribution to had. LbyL scattering



- Shaded blobs represent off-shell form factor $\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}$ where $\mathrm{PS} = \pi^0, \eta, \eta', \pi^{0\prime}, \ldots$
- Numerically dominant contribution to had. LbyL scattering
- Exchange of lightest state π^0 yields largest contribution $\longrightarrow$ warrants special attention
- Following Bijnens, Pallante, Prades (BPP) '95, '96; Hayakawa, Kinoshita, Sanda (HKS) '95, '96; Hayakawa, Kinoshita (HK) '98, we can define off-shell form-factor for π^0 as follows:

$$\int d^4x\, d^4y\, e^{i(q_1\cdot x + q_2\cdot y)}\, \langle 0|T\{j_\mu(x) j_\nu(y) P^3(0)\}|0\rangle$$

$$= \quad \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta\, \frac{i\langle\overline{\psi}\psi\rangle}{F_\pi}\, \frac{i}{(q_1+q_2)^2 - m_\pi^2}\, \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) + \ldots$$

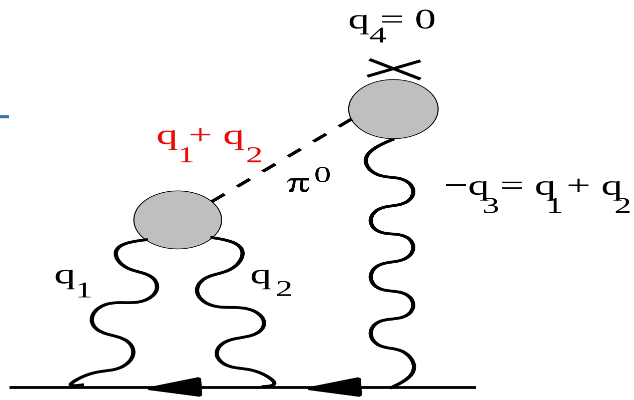
Up to small mixing effects of P^3 with η and η' and neglecting exchanges of heavier states like $\pi^{0\prime}, \pi^{0\prime\prime}, \ldots$

$j_\mu =$ light quark part of the electromagnetic current: $j_\mu(x) = (\overline{\psi}\hat{Q}\gamma_\mu\psi)(x)$, $\psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, $\hat{Q} = \mathrm{diag}(2, -1, -1)/3$

$P^3 = \overline{\psi} i\gamma_5 \frac{\lambda^3}{2}\psi = \left(\overline{u} i\gamma_5 u - \overline{d} i\gamma_5 d\right)/2$, $\langle\overline{\psi}\psi\rangle =$ single flavor quark condensate

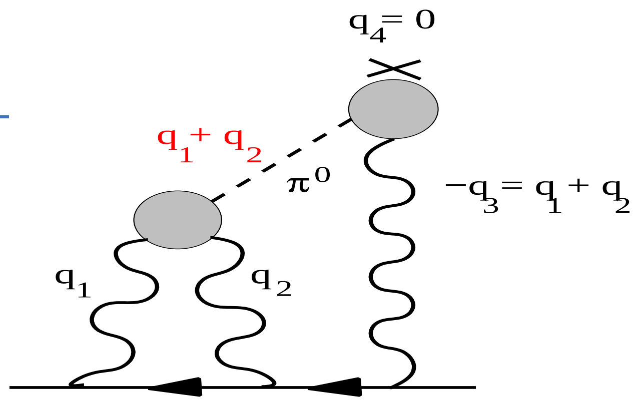
Off-shell versus on-shell form factors

- Off-shell form factors have been used to evaluate the pion-exchange contribution in BPP ’96, HKS ’96, HK ’98, but this seems to have been forgotten later. “Rediscovered” by Jegerlehner in ’07. Consider diagram:



$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

Off-shell versus on-shell form factors



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- On the other hand, Knecht + Nyffeler ’01, Bijnens + Persson ’01 used on-shell form factors:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$$

- But form factor at external vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$ violates momentum conservation, since momentum of external soft photon vanishes !
  Often the following misleading notation was used: $\mathcal{F}_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2, 0) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, (q_1+q_2)^2, 0)$

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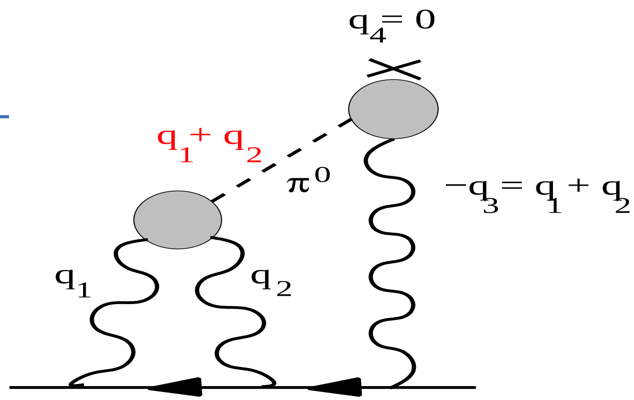

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

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- Melnikov + Vainshtein '03 had already observed this inconsistency and proposed to use

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, m_\pi^2, 0)$$

  i.e. a constant form factor at the external vertex given by the Wess-Zumino-Witten term

- However, this prescription will only yield the so-called pion-pole contribution and not the full pion-exchange contribution ! In general, off-shell form factors will enter at both vertices.

- Note: strictly speaking, the identification of the pion-exchange contribution is only possible, if the pion is on-shell. Only in some specific model where pions appear as propagating fields can one identify the contribution from off-shell pions.

New short-distance constraint on form factor at external vertex

Knecht + Nyffeler, EPJC '01: analysis of short-distance constraints (chiral limit, octet symmetry)

$$\langle V \underbrace{VP}_{\text{OPE}} \rangle \to \langle VT \rangle \qquad \text{Vector-Tensor two-point function}$$

$$\delta^{ab}(\Pi_{VT})_{\mu\rho\sigma}(p) = \int d^4x e^{ip\cdot x}\langle 0|T\{V_\mu^a(x)(\overline{\psi}\,\sigma_{\rho\sigma}\frac{\lambda^b}{2}\psi)(0)\}|0\rangle, \qquad \sigma_{\rho\sigma} = \frac{i}{2}[\gamma_\rho,\gamma_\sigma]$$

$$(\Pi_{VT})_{\mu\rho\sigma}(p) = (p_\rho\eta_{\mu\sigma} - p_\sigma\eta_{\mu\rho})\,\Pi_{VT}(p^2), \qquad \text{conservation of the vector current and parity invariance}$$

At external vertex in had. LbyL scattering the limit $p \to 0$ is relevant (soft photon) $\Rightarrow \Pi_{VT}(0)$

Ioffe + Smilga '84 defined quark condensate magnetic susceptibility χ of QCD in presence of constant external electromagnetic field:

$$\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F = e\,e_q\,\chi\,\langle\overline{\psi}\psi\rangle_0\,F_{\mu\nu}, \qquad e_u = 2/3,\ e_d = -1/3$$

Belyaev + Kogan '84 then showed that $\Pi_{VT}(0) = -\frac{\langle\overline{\psi}\psi\rangle_0}{2}\chi$

New short-distance constraint on the off-shell form factor at the external vertex (Nyffeler '09):

$$\lim_{\lambda\to\infty}\mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2,(\lambda q_1)^2,0) = -\frac{2}{3}\frac{F_0}{\langle\overline{\psi}\psi\rangle_0}\Pi_{VT}(0) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

$$= \frac{F_0}{3}\chi + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

- Note that there is no falloff in this limit, unless $\Pi_{VT}(0)$ vanishes !
- Corrections of $\mathcal{O}(\alpha_s)$ in OPE $\Rightarrow \chi$ depends on renormalization scale μ
- Unfortunately there is no agreement in the literature what the value of $\chi(\mu)$ should be !
  Range of values from $\chi(\mu \sim 0.5$ GeV$) \approx -9$ GeV^{-2} (Ioffe + Smilga '84; Vainshtein '03, Narison '08; ...)
  to $\chi(\mu \sim 1$ GeV$) \approx -3$ GeV^{-2} (Balitsky + Yung '83; Ball et al. '03; ...; Ioffe '09). Running with μ cannot explain such a difference.

New evaluation of pion-exchange contribution in large-N_C QCD

Framework: Minimal hadronic approximation for Green's function in large-N_C QCD (Peris et al. '98, ...)

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V)
- $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ fulfills all QCD short-distance (OPE) constraints
- Reproduces Brodsky-Lepage behavior (confirmed by CLEO data):

$$\lim_{Q^2\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$$

- Normalized to decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.6)$ eV

Off-shell LMD+V form factor (Knecht + Nyffeler, EPJC '01):

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}(q_3^2, q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2\, q_2^2\, (q_1^2 + q_2^2 + q_3^2) + P_H^V(q_1^2, q_2^2, q_3^2)}{(q_1^2 - M_{V_1}^2)\,(q_1^2 - M_{V_2}^2)\,(q_2^2 - M_{V_1}^2)\,(q_2^2 - M_{V_2}^2)}$$

$$P_H^V(q_1^2, q_2^2, q_3^2) = h_1\,(q_1^2 + q_2^2)^2 + h_2\, q_1^2\, q_2^2 + h_3\,(q_1^2 + q_2^2)\, q_3^2 + h_4\, q_3^4 + h_5\,(q_1^2 + q_2^2) + h_6\, q_3^2 + h_7, \qquad q_3^2 = (q_1 + q_2)^2$$

$F_\pi = 92.4$ MeV, $M_{V_1} = M_\rho = 775.49$ MeV, $M_{V_2} = M_{\rho'} = 1.465$ GeV

We view our evaluation as being a part of a full calculation of the hadronic light-by-light scattering contribution using a resonance Lagrangian along the lines of the Resonance Chiral Theory (Ecker et al. '89, ...), which also fulfills all the relevant QCD short-distance constraints.

Fixing the LMD+V model parameters h_i

h_1, h_2, h_5, h_7 are quite well known:

- $h_1 = 0$ GeV^2 (Brodsky-Lepage behavior $\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$)
- $h_2 = -10.63$ GeV^2 (Melnikov + Vainshtein '03: Higher twist corrections in OPE)
- $h_5 = 6.93 \pm 0.26$ GeV$^4 - h_3 m_\pi^2$ (fit to CLEO data of $\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0)$)
- $h_7 = -N_C M_{V_1}^4 M_{V_2}^4 / (4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$
  $= -14.83$ GeV$^6 - h_6 m_\pi^2 - h_4 m_\pi^4$ (normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

h_3, h_4, h_6 are unknown / less constrained:

- New short-distance constraint $\Rightarrow h_1 + h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$ $(*)$

  LMD ansatz for $\langle VT \rangle \Rightarrow \chi^{\mathrm{LMD}} = -2/M_V^2 = -3.3$ GeV^{-2} (Balitsky + Yung '83)

  Close to $\chi(\mu = 1$ GeV$) = -(3.15 \pm 0.30)$ GeV^{-2} (Ball et al. '03)

  Assume large-N_C (LMD/LMD+V) framework is self-consistent

  $\Rightarrow \chi = -(3.3 \pm 1.1)$ GeV^{-2}

  $\Rightarrow$ vary $h_3 = (0 \pm 10)$ GeV^2 and determine h_4 from relation $(*)$ and vice versa

- Final result for $a_\mu^{\mathrm{LbyL};\pi^0}$ is very sensitive to h_6

  Assume that LMD/LMD+V estimates of low-energy constants from chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$ are self-consistent.
  Assume 100% error on estimate for the relevant, presumably small low-energy constant.
  $\Rightarrow h_6 = (5 \pm 5)$ GeV^4

Result for pseudoscalar-exchange contribution

- π^0
  - Our new estimate (Nyffeler '09; Jegerlehner + Nyffeler '09):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

    With off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}$ which obeys new short-distance constraint.
  - Largest uncertainty from $h_6 = (5 \pm 5)$ GeV^4 $\Rightarrow \pm 6.4 \times 10^{-11}$ in $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$

    If we would vary $h_6 = (0 \pm 10)$ GeV^4 $\Rightarrow \pm 12 \times 10^{-11}$!
  - Varying $\chi = -(3.3 \pm 1.1)$ GeV^{-2} $\Rightarrow \pm 2.1 \times 10^{-11}$

    Exact value of χ not that important, but range does not include Vainshtein's estimate $\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$ GeV^{-2}
  - Varying $h_3 = (0 \pm 10)$ GeV^2 $\Rightarrow \pm 2.5 \times 10^{-11}$ (h_4 via $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$)
  - Added errors linearly.

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  - Added errors linearly.
- η, η'
  - Short-distance analysis of LMD+V form factor in Knecht + Nyffeler, EPJC '01, performed in chiral limit and assuming octet symmetry $\Rightarrow$ not valid anymore for η and η' !
  - Simplified approach: VMD form factors normalized to decay width $\Gamma(\mathrm{PS} \longrightarrow \gamma\gamma)$.

$$\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}^{\mathrm{VMD}}(q_3^2, q_1^2, q_2^2) = -\frac{N_c}{12\pi^2 F_{\mathrm{PS}}} \frac{M_V^2}{(q_1^2 - M_V^2)} \frac{M_V^2}{(q_2^2 - M_V^2)}, \quad \mathrm{PS} = \eta, \eta'$$

  - $\Rightarrow a_\mu^{\mathrm{LbyL};\eta} = 14.5 \times 10^{-11}$ and $a_\mu^{\mathrm{LbyL};\eta'} = 12.5 \times 10^{-11}$

    Not taking pole-approximation as done in Melnikov + Vainshtein '03 !

    Note: VMD form factor has too strong damping at large momenta $\longrightarrow$ values might be a bit too small !

Our estimate for the sum of all light pseudoscalars (Nyffeler '09; Jegerlehner + Nyffeler '09):

$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$$

Pseudoscalar exchanges: results in the literature

Model for $\mathcal{F}_{P^{(*)}\gamma^*\gamma^*}$	$a_\mu(\pi^0) \times 10^{11}$	$a_\mu(\pi^0, \eta, \eta') \times 10^{11}$
modified ENJL (off-shell) [BPP]	59(9)	85(13)
VMD / HLS (off-shell) [HKS,HK]	57(4)	83(6)
LMD+V (on-shell, $h_2 = 0$) [KN]	58(10)	83(12)
LMD+V (on-shell, $h_2 = -10$ GeV^2) [KN]	63(10)	88(12)
LMD+V (on-shell, constant FF at ext. vertex) [MV]	77(7)	114(10)
nonlocal χQM (off-shell) [DB]	65(2)	—
LMD+V (off-shell) [N]	72(12)	99(16)
AdS/QCD (on-shell) [HoK]	68	102
[PdRV]	—	114(13)
[JN]	72(12)	99(16)

BPP = Bijnens, Pallante, Prades '95, '96, '02 (ENJL = Extended Nambu-Jona-Lasinio model); HK(S) = Hayakawa, Kinoshita, Sanda '95, '96; Hayakawa, Kinoshita '98, '02 (HLS = Hidden Local Symmetry model); KN = Knecht, Nyffeler '01; MV = Melnikov, Vainshtein '03; DB = Dorokhov, Broniowski '08 (χQM = Chiral Quark Model); N = Nyffeler '09; HoK = Hong, Kim '09; PdRV = Prades, de Rafael, Vainshtein '09; JN = Jegerlehner, Nyffeler '09

- BPP use rescaled VMD result for η, η'. Also all LMD+V evaluations use VMD for η, η' !
- Off-shell form factors used in BPP, HKS presumably do not fulfill new short-distance constraint at external vertex and might have too strong damping $\longrightarrow$ smaller values.
- Our result for pion with off-shell form factors at both vertices is not too far from value given by M+V '03, but this is pure coincidence ! Approaches not comparable ! M+V '03 evaluate pion-pole contribution and use on-shell form factors (constant form factor at external vertex).

  Note: Following M+V '03 and using $h_2 = -10$ GeV^2 we obtain 79.8×10^{-11} for the pion-pole contribution, close to the value 79.6×10^{-11} given in Bijnens + Prades '07 and 79.7×10^{-11} in D+B '08

Conclusions

- Jegerlehner '07: one should use off-shell form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ to evaluate pion-exchange contribution. As done in earlier papers by BPP, HKS, HK !
Prescription by Melnikov + Vainshtein '03 to use a constant (WZW) form factor at the external vertex only yields pion-pole contribution with on-shell form factors $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$.
- We derived a new short-distance constraint on off-shell form factor at external vertex:

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1)^2, (\lambda q_1)^2, 0) = \frac{F_0}{3}\chi + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

[$\chi =$ chiral cond. mag. susceptibility]

- We newly evaluated pion-exchange contribution within large-N_C approximation using off-shell LMD+V form factor that fulfills all QCD short-distance constraints:

$$a_\mu^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

[BPP: 59 ± 9; HKS: 57 ± 4; KN: 58 ± 10; MV: 77 ± 7 in units of 10^{-11}]

- Updated values for η and η' (using simple VMD form factors):

$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$$

[BPP: 85 ± 13; HKS: 83 ± 6; KN: 83 ± 12; MV: 114 ± 10 in units of 10^{-11}]

- Combined with evaluations of the other contributions we get:

$$a_\mu^{\mathrm{LbyL;had}} = (116 \pm 40) \times 10^{-11}$$

[PdRV: $(105 \pm 26) \times 10^{-11}$]

- Corresponding contributions for the electron (Nyffeler '09, Jegerlehner + Nyffeler '09):

$$a_e^{\mathrm{LbyL};\pi^0} = (2.98 \pm 0.34) \times 10^{-14}, \quad a_e^{\mathrm{LbyL};\eta} = 0.49 \times 10^{-14}, \quad a_e^{\mathrm{LbyL};\eta'} = 0.39 \times 10^{-14}$$

$$a_e^{\mathrm{LbyL;PS}} = (3.9 \pm 0.5) \times 10^{-14}$$

$$a_e^{\mathrm{LbyL;had}} = (3.9 \pm 1.3) \times 10^{-14}$$

[Guesstimate ! Jegerlehner + Nyffeler '09]

Note: naive rescaling would yield a too small result: $a_e^{\mathrm{LbyL};\pi^0}(\mathrm{rescaled}) = (m_e/m_\mu)^2\, a_\mu^{\mathrm{LbyL};\pi^0} = 1.7 \times 10^{-14}$!

Backup slides

Estimates for the quark condensate magnetic susceptibility χ

Authors	Method	$\chi(\mu)$ [GeV]$^{-2}$	Footnote
Ioffe + Smilga '84	QCD sum rules	$\chi(\mu = 0.5\ \text{GeV}) = -\left(8.16^{+2.95}_{-1.91}\right)$	[1]
Narison '08	QCD sum rules	$\chi = -(8.5 \pm 1.0)$	[2]
Vainshtein '03	OPE for $\langle VVA \rangle$	$\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$	[3]
Gorsky + Krikun '09	AdS/QCD	$\chi = -(2.15 N_C)/(8\pi^2 F_\pi^2) = -9.6$	[4]
Dorokhov '05	Instanton liquid model	$\chi(\mu \sim 0.5 - 0.6\ \text{GeV}) = -4.32$	[5]
Ioffe '09	Zero-modes of Dirac operator	$\chi(\mu \sim 1\ \text{GeV}) = -3.52\ (\pm 30 - 50\%)$	[6]
Buividovich et al. '09	Lattice	$\chi = -1.547(6)$	[7]
Balitsky + Yung '83	LMD for $\langle VT \rangle$	$\chi = -2/M_V^2 = -3.3$	[8]
Belyaev + Kogan '84	QCD sum rules for $\langle VT \rangle$	$\chi(0.5\ \text{GeV}) = -(5.7 \pm 0.6)$	[9]
Balitsky et al. '85	QCD sum rules for $\langle VT \rangle$	$\chi(1\ \text{GeV}) = -(4.4 \pm 0.4)$	[9]
Ball et al. '03	QCD sum rules for $\langle VT \rangle$	$\chi(1\ \text{GeV}) = -(3.15 \pm 0.30)$	[9]

[1]: QCD sum rule evalation of nucleon magnetic moments.
[2]: Recent reanalysis of these sum rules for nucleon magnetic moments. At which scale μ ?
[3]: Probably at low scale $\mu \sim 0.5$ GeV, since pion dominance was assumed in derivation.
[4]: From derivation in holographic model it is not clear what is the relevant scale μ.
[5]: The scale is set by the inverse average instanton size ρ^{-1}.
[6]: Study of zero-mode solutions of Dirac equation in presence of arbitrary gluon fields (à la Banks-Casher).
[7]: Again à la Banks-Casher. Quenched lattice calculation for $SU(2)$. μ dependence is not taken into account. Lattice spacing corresponds to 2 GeV.
[8]: The leading short-distance behavior of Π_{VT} is given by (Craigie + Stern '81)

$$\lim_{\lambda \to \infty} \Pi_{VT}((\lambda p)^2) = -\frac{1}{\lambda^2} \frac{\langle \overline{\psi}\psi \rangle_0}{p^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

Assuming that the two-point function Π_{VT} is well described by the multiplet of the lowest-lying vector mesons (LMD) and satisfies this OPE constraint leads to the ansatz (Balitsky + Yung '83, Belyaev + Kogan '84, Knecht + Nyffeler, EPJC '01)

$$\Pi_{VT}^{\text{LMD}}(p^2) = -\langle \overline{\psi}\psi \rangle_0 \frac{1}{p^2 - M_V^2} \quad \Rightarrow \chi^{\text{LMD}} = -\frac{2}{M_V^2} = -3.3\ \text{GeV}^{-2}$$

Not obvious at which scale. Maybe $\mu = M_V$ as for low-energy constants in ChPT.
[9]: LMD estimate later improved by taking more resonance states $\rho', \rho'', \ldots$ in QCD sum rule analysis of $\langle VT \rangle$.
Note that the last value by Ball et al. is very close to original LMD estimate !

Constraining the LMD+V model parameter h_6

- Final result for $a_\mu^{\mathrm{LbyL};\pi^0}$ is very sensitive to value of h_6. We can get some indirect information on size and sign of h_6 as follows.
- Estimates of low-energy constants in chiral Lagrangians via exchange of resonances work quite well. However, we may get some corrections, if we consider the exchange of heavier resonances as well. Typically, a large-N_C error of 30% can be expected.
- In $\langle VVP \rangle$ appear 2 combinations of low-energy constants from the chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$, denoted by A_{V,p^2} and $A_{V,(p+q)^2}$ in Knecht + Nyffeler, EPJC '01.

$$A_{V,p^2}^{\mathrm{LMD}} = \frac{F_\pi^2}{8M_V^4} - \frac{N_C}{32\pi^2 M_V^2} = -1.11 \, \frac{10^{-4}}{F_\pi^2}$$

$$A_{V,p^2}^{\mathrm{LMD+V}} = \frac{F_\pi^2}{8M_{V_1}^4} \frac{h_5}{M_{V_2}^4} - \frac{N_C}{32\pi^2 M_{V_1}^2} \left(1 + \frac{M_{V_1}^2}{M_{V_2}^2} \right) = -1.36 \, \frac{10^{-4}}{F_\pi^2}$$

The relative change is only about 20%, well within expected large-N_C uncertainty !

$$A_{V,(p+q)^2}^{\mathrm{LMD}} = -\frac{F_\pi^2}{8M_V^4} = -0.26 \, \frac{10^{-4}}{F_\pi^2}, \qquad A_{V,(p+q)^2}^{\mathrm{LMD+V}} = -\frac{F_\pi^2}{8M_{V_1}^4 M_{V_2}^4} h_6$$

Note that $A_{V,(p+q)^2}^{\mathrm{LMD}}$ is "small" compared to A_{V,p^2}^{LMD}. About same size as absolute value of the shift in A_{V,p^2} when going from LMD to LMD+V !

- Assuming that LMD/LMD+V framework is self-consistent, but allowing for a 100% uncertainty of $A_{V,(p+q)^2}^{\mathrm{LMD}}$, we get the range $h_6 = (5 \pm 5)$ GeV^4

Further results concerning the pion-exchange contribution

$a_\mu^{\mathrm{LbyL};\pi^0} \times 10^{11}$ with the off-shell LMD+V form factor:

	$h_6 = 0$ GeV^4	$h_6 = 5$ GeV^4	$h_6 = 10$ GeV^4
$h_3 = -10$ GeV^2	68.4	74.1	80.2
$h_3 = 0$ GeV^2	66.4	71.9	77.8
$h_3 = 10$ GeV^2	64.4	69.7	75.4
$h_4 = -10$ GeV^2	65.3	70.7	76.4
$h_4 = 0$ GeV^2	67.3	72.8	78.8
$h_4 = 10$ GeV^2	69.2	75.0	81.2

$\chi = -3.3$ GeV^{-2}, $h_1 = 0$ GeV^2, $h_2 = -10.63$ GeV^2 and $h_5 = 6.93$ GeV$^4 - h_3 m_\pi^2$

When varying h_3 (upper half of table), h_4 is fixed by constraint $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$. In the lower half the procedure is reversed.

Within scanned region: Minimal value: 63.2×10^{-11} [$\chi = -2.2$ GeV^{-2}, $h_3 = 10$ GeV^2, $h_6 = 0$ GeV^4]

Maximum value: 83.3×10^{-11} [$\chi = -4.4$ GeV^{-2}, $h_4 = 10$ GeV^2, $h_6 = 10$ GeV^4]

Take average of results for $h_6 = 5$ GeV^4 for $h_3 = 0$ GeV^2 and $h_4 = 0$ GeV^2 as estimate: $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$

Added errors from χ, h_3 (or h_4) and h_6 linearly. Do not follow Gaussian distribution !

Parametrization of $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$ for arbitrary model parameters h_i

The h_i enter the LMD+V form factor linearly in the numerator, therefore (Nyffeler '09):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \left[\sum_{i=1}^{7} c_i\, \tilde{h}_i + \sum_{i=1}^{7}\sum_{j=i}^{7} c_{ij}\, \tilde{h}_i\, \tilde{h}_j\right]$$

with dimensionless coefficients $c_i, c_{ij} \sim 10^{-4}$, if we measure the h_i in appropriate units of GeV $\longrightarrow \tilde{h}_i$ (see Nyffeler '09 for the values)

h_1, h_3, h_4 not independent, but must obey the relation $h_1 + h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$, because of the new short-distance constraint.

h_1, h_2, h_5, h_7 are quite well known $\longrightarrow$ can write down a simplified expression with only h_3, h_4, h_6 as free parameters (up to constraint):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \Big[503.3764 - 6.5223\, \tilde{h}_3 - 5.0962\, \tilde{h}_4 + 7.8557\, \tilde{h}_6 + 0.3017\, \tilde{h}_3^2 + 0.5683\, \tilde{h}_3\, \tilde{h}_4$$
$$-0.1747\, \tilde{h}_3\, \tilde{h}_6 + 0.2672\, \tilde{h}_4^2 - 0.1411\, \tilde{h}_4\, \tilde{h}_6 + 0.0642\, \tilde{h}_6^2 \Big] \times 10^{-4}$$

Hadronic light-by-light scattering in the muon $g-2$

Some selected results for the various contributions to $a_\mu^{\mathrm{LbyL;had}} \times 10^{11}$:

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	$-$	114 ± 13	99 ± 16
axial vectors	2.5 ± 1.0	1.7 ± 1.7	$-$	22 ± 5	$-$	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	$-$	$-$	$-$	$-$	-7 ± 7	-7 ± 2
π, K loops	-19 ± 13	-4.5 ± 8.1	$-$	$-$	$-$	-19 ± 19	-19 ± 13
π, K loops $+$subl. N_C	$-$	$-$	$-$	0 ± 10	$-$	$-$	$-$
quark loops	21 ± 3	9.7 ± 11.1	$-$	$-$	$-$	2.3	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, Nyffeler '01; MV = Melnikov, Vainshtein '03; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = Nyffeler '09, JN = Jegerlehner, Nyffeler '09

- Pseudoscalar-exchange contribution dominates numerically. But other contributions are not negligible. Note cancellation between π, K-loops and quark loops !
- $(80 \pm 40) \times 10^{-11}$ not in KN '01; estimate used by Marseille group before MV '03.
- PdRV: Do not consider dressed light quark loops as separate contribution ! Assume it is already taken into account by using short-distance constraint of MV '03 on pseudoscalar-pole contribution. Why should this be the case ?
  Added all errors in quadrature ! Like HK(S). Too optimistic ?
- N, JN: Evaluation of the axial vectors by MV '03 is definitely some improvement over earlier calculations. It seems, however, again to be only the axial-vector pole contribution.
  Added all errors linearly. Like BPP, MV, BP, MdRR. Too pessimistic ?