Pion Polarizability: no Conflict between Dispersion Theory and ChPT

Barbara Pasquini[1], Dieter Drechsel[2], and Stefan Scherer[2]

[1]Università Pavia and INFN

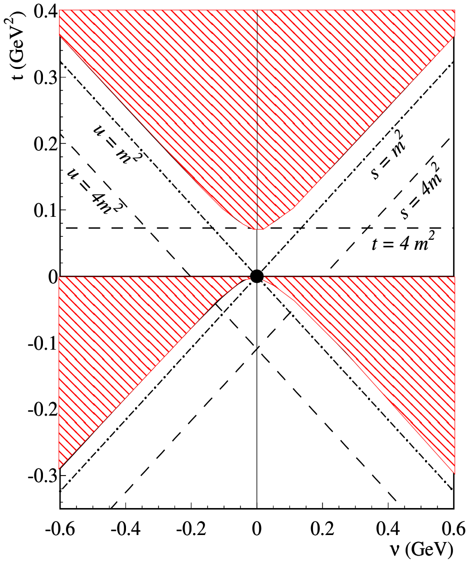
[2]Institut für Kernphysik , Universität Mainz

- PDS: B. Pasquini, D.Drechsel, S.Scherer, Phys. Rev. C 77, 065211 (2008)
- FK: L.V. Fil'kov, V.L. Kashevarov, Phys. Rev. C 72, 035211 (2005)
- GIS: J. Gasser, M.A. Ivanov, M.E. Sainio, Nucl. Phys. B 745, 84 (2006)

Introduction

- $\alpha =$ electric polarizability, $\beta =$ magnetic polarizability
  units: $10^{-4}\mathrm{fm}^3$
  forward polarizability of pion, $\alpha + \beta$
  backward polarizability of pion, $\alpha - \beta$
- Prediction of ChPT at $\mathcal{O}(p^6)$ [GIS]:
  $\alpha_{\pi^+} + \beta_{\pi^+} = 0.16 \pm 0.1, \quad \alpha_{\pi^+} - \beta_{\pi^+} = 5.7 \pm 1.0$
- Prediction of Ref. [FK]:
  $\alpha_{\pi^+} + \beta_{\pi^+} = 0.17 \pm 0.02, \quad \alpha_{\pi^+} - \beta_{\pi^+} = 13.60 \pm 2.15$
- experimental data: next talk by J. Friedrich!

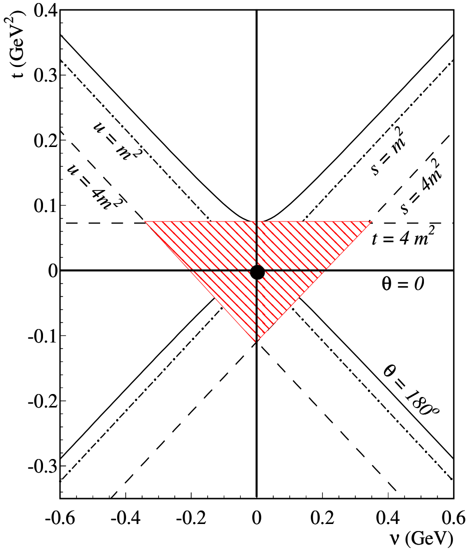
Physical regions in Mandelstam plane



- variables s, u, t with $s + u + t = 2m^2$
- rectangular coordinates $\nu = (s - u)/4m$ (energy) t (momentum transfer)
- red hatched: physical regions ($\gamma + \pi \to \gamma + \pi$, $\gamma + \gamma \to \pi + \pi$)
- two-pion thresholds at $s = 4m^2$, $u = 4m^2$, $t = 4m^2$
- integration paths for DRs: $t = 0$ (forward DR), $\theta = 180^\circ$ (backward DR), $u = m^2$, $s = m^2$, etc.

polarizability determined at $s = u = m^2$ or $\nu = t = 0$

Taylor series in circle about $\nu = t = 0$



- There should be no singularities in red hatched triangle.
- Within a circle about origin of Mandelstam plane and fitting into triangle, amplitude given by Taylor series.
- Model of L.V. Fil'kov and V.L. Kashevarov has singularities $1/\sqrt{s}$, $1/\sqrt{t}$, $1/\sqrt{u}$ (Xing symmetry).

FK model for vector meson contributions

$$M^{+-}(s) = \frac{4\,g(s)^2}{M^2 - s - iM\Gamma(s)}, \quad M^{++}(s) = -s\,M^{+-}(s) \qquad (1)$$

with energy-dependent width (P wave $\sim q^3$ at threshold)

$$\Gamma(s) = \left(\frac{s - 4\,m^2}{M^2 - 4\,m^2}\right)^{3/2} \Gamma_0 \qquad (2)$$

with Γ_0 width of vector meson V at resonance, $s = M^2$, and energy-dependent coupling constant ($\sim s^{-1/2}$ singularity at $s = 0$)

$$g(s)^2 = \frac{6\pi M}{s^{1/2}} \left(\frac{M}{M^2 - m^2}\right)^3 \Gamma_\gamma\,, \qquad (3)$$

with partial decay width Γ_γ for $V \to \pi + \gamma$.

Polarizabilities for FK model

$$\alpha+\beta=\frac{m}{2\pi}M^{+-}(s=m^2),\quad \alpha-\beta=\frac{1}{2\pi m}M^{++}(s=m^2) \qquad (4)$$

From Eq. (1) $\rightarrow M^{++}(s=m^2)=-m^2M^{+-}(s=m^2)\Rightarrow$

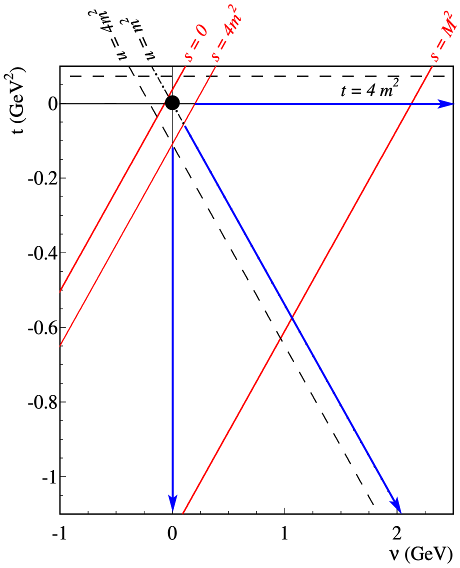
$$\alpha+\beta=-(\alpha-\beta)\quad\Rightarrow\quad\alpha=0 \qquad (5)$$

Transition pion ($J^P=0^-$) to intermediate vector meson ($J^P=1^-$) is magnetic dipole transition, should yield paramagnetic contribution to β.

Table: Polarizabilities according to Ref. FK, in units of $10^{-4}\,\mathrm{fm}^3$

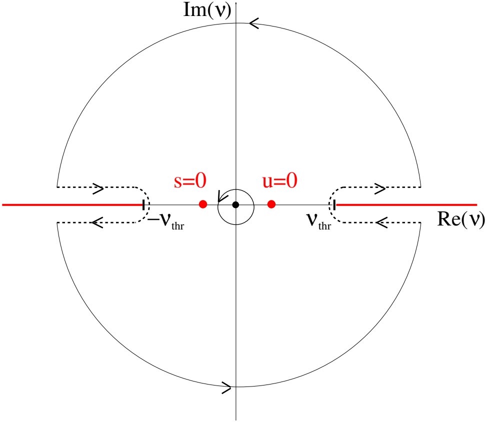
meson	$\alpha+\beta$	$\alpha-\beta$	α	β
ρ	0.063	-1.15	-0.54	0.61
ω	0.721	-12.56	-5.92	6.64

Singularities of FK model



- Model has unphysical singularities at $s = 0$, $t = 0$, $u = 0$, very close to $s = u = m^2$ or $\nu = t = 0$. These singularities are prerequisite to fulfill Titchmarsh theorem (square integrability $\rightarrow$ Hilbert transforms).
- FK evaluate dispersion integral over **physical cut** only. Imaginary parts "below physical threshold" set zero. Result: Non-analytic function, Titchmarsh theorem annulled.

Contour for forward DR

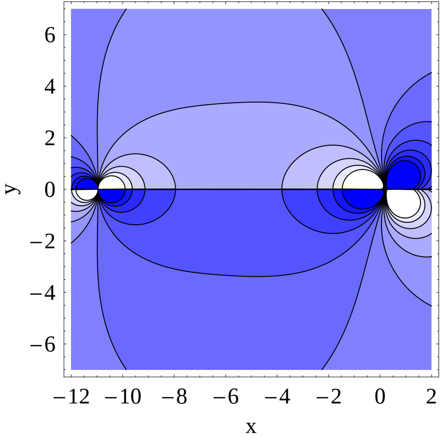


- Step 1: Contour integral along small circle with radius ε about $\nu = 0$: $F(0) = 1/(2\pi i) \oint F(\nu')/\nu'$ (Cauchy integral)
- Step 2: Extend contour until it hits non-analytic structures or infinity.
- Results $\Rightarrow$

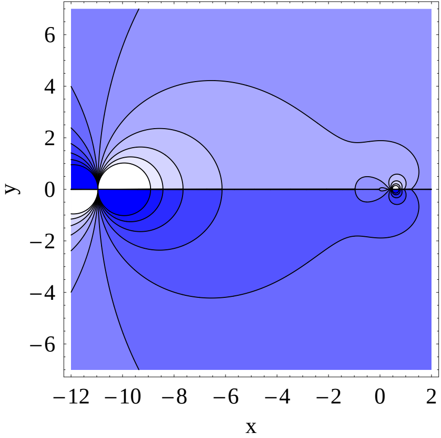
- No contribution from "big circle" $R \to \infty$.
- Physical cuts $-\infty < \mathrm{Re}(\nu) \leq -\nu_{\mathrm{thr}}$ and $\nu_{\mathrm{thr}} \leq \mathrm{Re}(\nu) < \infty$.
- Unphysical cuts starting at $s = 0$ and $u = 0$ yield large contributions. FK ignore respective imaginary parts, as a result the amplitude becomes non-analytic in parts of the complex plane.

Contour plots for FK model

$\mathrm{Im}[M^{+-}(s = x + iy)]$ $(\Rightarrow \alpha + \beta)$



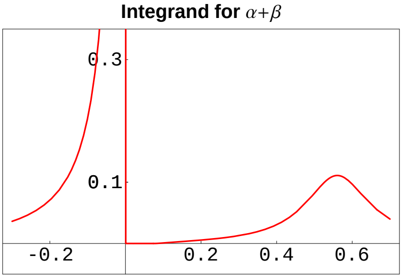
$\mathrm{Im}[M^{++}(s = x + iy)]$ $(\Rightarrow \alpha - \beta)$



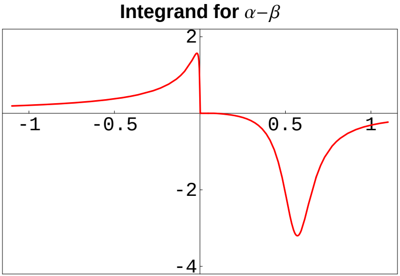
- Physical cut with maximum near $x = M^2 \approx 0.55\mathrm{GeV}^2$.
- Unphysical (left-hand) cut with "bound state" near $-11\ \mathrm{GeV}^2$.
- $\alpha \pm \beta$ determined at $\{x = m^2, y = 0\}$, "squeezed in between cuts".

Integrands for $\alpha \pm \beta$

as function of $x = \mathrm{Re}(s)$ in units GeV^2



$$\alpha + \beta = 0.17 \ (\mathrm{l.c.}) + 0.03 \ (\mathrm{r.c.})$$



$$\alpha - \beta = 0.98 \ (\mathrm{l.c.}) - 1.18 \ (\mathrm{r.c.})$$

Contributions to electric (α) and magnetic (β) polarizabilities

for ρ meson with several resonance models

A = pole at $M - i\Gamma_0/2$, B = pole at $M - i\Gamma(s)/2$, C = $\Gamma(s)^2 \to 0$
$\{A0, B0, C0\}$ with $g(M^2)$ and $\{A, B, C\}$ with $g(s)$

$\alpha + \beta$

	real	r.c.	l.c.	rest
$A0$	0.04	0.04	0.00	–
$B0$	0.04	0.03	–	0.01
$C0$	0.04	0.03	–	0.00
A	0.20	0.05	0.15	–
B	0.20	0.03	0.17	0.00
C	0.20	0.03	0.17	–

$\alpha - \beta$

	real	r.c.	l.c.	rest
$A0$	−0.04	−1.04	−0.08	1.08
$B0$	−0.04	−1.15	–	1.11
$C0$	−0.04	−1.93	–	1.89
A	−0.20	−1.06	0.86	–
B	−0.20	−1.02	0.81	0.01
C	−0.20	−1.18	0.98	–

Numbers in units of 10^{-4} fm^3. First column: Model, second column: directly from real part of amplitudes, further columns: contributions from Cauchy integral (r.c. and l.c. = integrals along right and left cuts, rest= residues of poles and "big circle"). Note: Sum of dispersive contributions ≡ real part.

Conclusions

- **Titchmarsh theorem**: **IF** $f(\nu) = f(x + iy)$ analytic in complex plane, except for isolated singular points on real axis, and $\int_{-\infty}^{\infty} |f(x+iy)|^2$ finite for any $|y| > 0$, **THEN** $\mathrm{Re(f)}$ and $\mathrm{Im(f)}$ are Hilbert transforms (related by dispersion relations).
- FK model fulfills integrability only because of energy dependent width and coupling constant.

Conclusions

- **Titchmarsh theorem**: **IF** $f(\nu) = f(x+iy)$ analytic in complex plane, except for isolated singular points on real axis, and $\int_{-\infty}^{\infty} |f(x+iy)|^2$ finite for any $|y| > 0$, **THEN** $\mathrm{Re(f)}$ and $\mathrm{Im(f)}$ are Hilbert transforms (related by dispersion relations).
- FK model fulfills integrability only because of energy dependent width and coupling constant.
- **Analyticity**: $\lim_{\nu' \to \nu} \frac{f(\nu') - f(\nu)}{\nu' - \nu} \to f'(\nu)$ derivative independent of path $\nu' \to \nu$, except for isolated singular points. Setting $\mathrm{Im}(f(\nu)) \Rightarrow 0$ within region G makes G a non-analytic region.

Conclusions

- **Titchmarsh theorem**: **IF** $f(\nu) = f(x+iy)$ analytic in complex plane, except for isolated singular points on real axis, and $\int_{-\infty}^{\infty} |f(x+iy)|^2$ finite for any $|y| > 0$, **THEN** $\mathrm{Re(f)}$ and $\mathrm{Im(f)}$ are Hilbert transforms (related by dispersion relations).
- FK model fulfills integrability only because of energy dependent width and coupling constant.
- **Analyticity**: $\lim_{\nu' \to \nu} \frac{f(\nu') - f(\nu)}{\nu' - \nu} \to f'(\nu)$
  derivative independent of path $\nu' \to \nu$, except for isolated singular points. Setting $\mathrm{Im}(f(\nu)) \Rightarrow 0$ within region G makes G a non-analytic region.
- It is inconsistent to introduce energy-dependent width for convergence at large energies (Titchmarsh!) and later ignore consequences (unphysical cut) at small energies (annul Titchmarsh, create non-analytic region!).

Conclusions

- **Titchmarsh theorem**: **IF** $f(\nu) = f(x+iy)$ analytic in complex plane, except for isolated singular points on real axis, and $\int_{-\infty}^{\infty} |f(x+iy)|^2$ finite for any $|y| > 0$, **THEN** $\mathrm{Re(f)}$ and $\mathrm{Im(f)}$ are Hilbert transforms (related by dispersion relations).
- FK model fulfills integrability only because of energy dependent width and coupling constant.
- **Analyticity**: $\lim_{\nu' \to \nu} \frac{f(\nu')-f(\nu)}{\nu'-\nu} \to f'(\nu)$ derivative independent of path $\nu' \to \nu$, except for isolated singular points. Setting $\mathrm{Im}(f(\nu)) \Rightarrow 0$ within region G makes G a non-analytic region.
- It is inconsistent to introduce energy-dependent width for convergence at large energies (Titchmarsh!) and later ignore consequences (unphysical cut) at small energies (annul Titchmarsh, create non-analytic region!).
- Inconsistency introduces strong and negative electric polarizability from magnetic dipole transition $(\pi \to \rho, \omega, \ldots)$.
- σ meson exchange in t-channel: $\alpha - \beta \to \infty$ because of $1/\sqrt{t}$.