CURRENT ALGEBRA

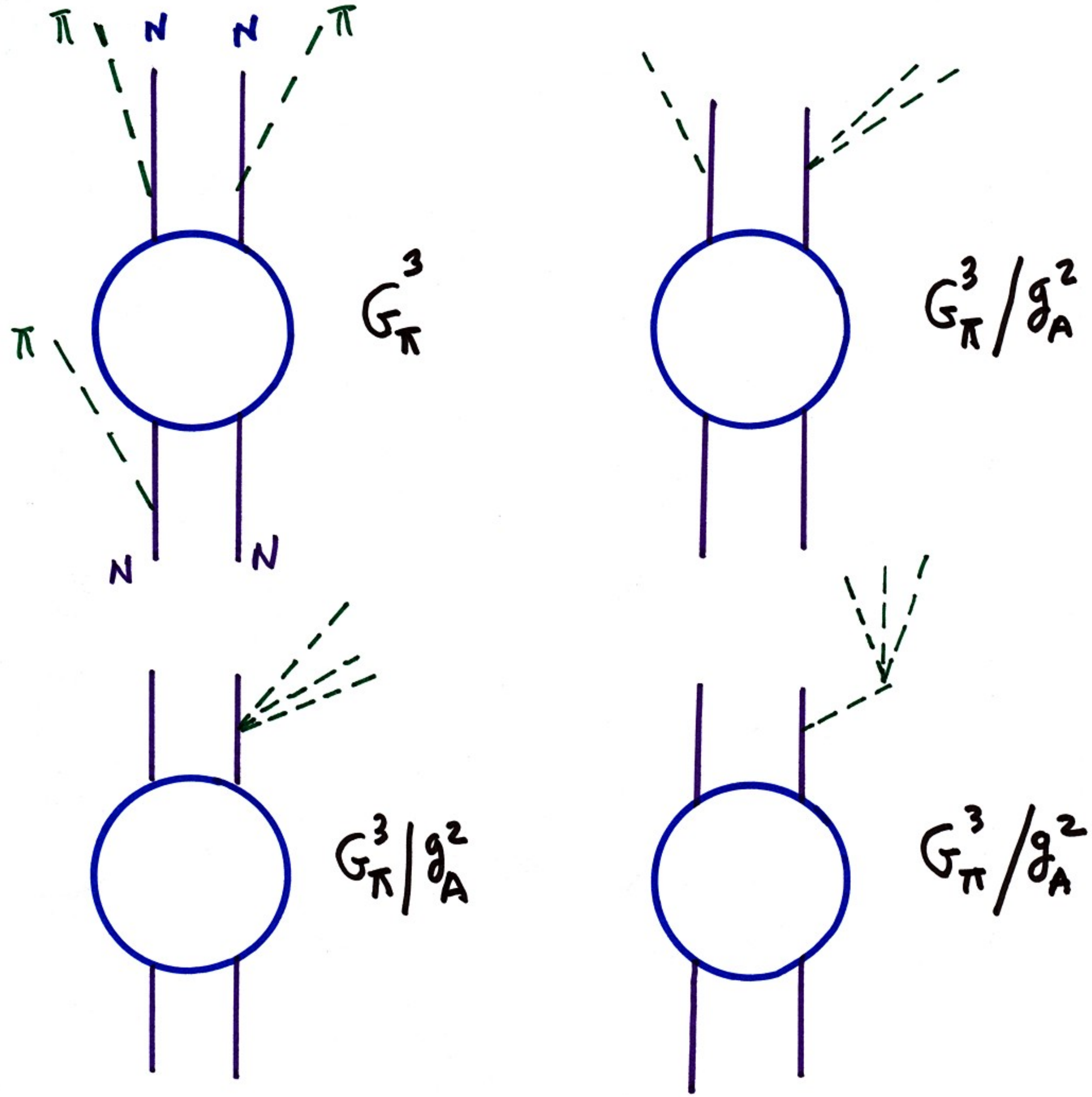
- $\partial_\mu A^\mu = 0$ for $m_\pi = 0$
  $\Rightarrow F_\pi = 2m_N g_A / G_\pi$ Nambu 1960
- π Goldstone boson
  Goldstone 1961; Nambu 1961;
  Goldstone, Salam, SW 1962
- Single pion emission
  Nambu & Shrauner/Lurie 1962
- Current commutators Gell-Mann 1964
- g_A sum rule Adler, Weisberger, 1965

$$SU(2) \times SU(2) \quad \Big(\equiv SO(4) \Big)$$

Strong Interactions:

- $\pi N \to \pi N$ Tomozawa, SW, 1966
- $\pi\pi \to \pi\pi$ SW 1966

Multiple Pion Emission SW 1965



Linear σ-Model

$$\mathcal{L} = -\frac{1}{2}\left[\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + \partial_\mu\sigma\,\partial^\mu\sigma\right]$$
$$-\frac{m^2}{2}\left(\sigma^2+\vec{\pi}^2\right) - \frac{\lambda}{4}\left(\sigma^2+\vec{\pi}^2\right)^2$$
$$-\bar{N}\gamma^\mu\partial_\mu N - G_\pi\bar{N}\left(\sigma + 2i\gamma_5\vec{\pi}\cdot\vec{t}\right)N$$
$$-c\sigma$$

Transform away $\vec{\pi}$:

$$\left(\vec{\pi},\sigma\right)\mapsto\left(0,\sigma'\right), \qquad \sigma' = \sqrt{\sigma^2+\vec{\pi}^2}$$

$SO(4)$ rotation angle:

$$\vec{\pi}' \equiv F\vec{\pi}/[\sigma+\sigma'], \qquad F\equiv\sqrt{-m^2/\Lambda}$$

Throw away $SO(4)$ scalar mode:

$$\sigma'\mapsto F/2$$

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{2}\left(1+\frac{\vec{\pi}^2}{F^2}\right)^{-2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} \\
& -\bar{N}\Bigg[\gamma^\mu\partial_\mu + G_\pi F/2 \\
& \qquad + i\gamma^\mu\left(1+\frac{\vec{\pi}^2}{F^2}\right)^{-1}\Big[2\gamma_5\vec{t}\cdot\partial_\mu\vec{\pi}/F \\
& \qquad\quad + 2\vec{t}\cdot(\vec{\pi}\times\partial_\mu\vec{\pi})/F^2\Big]\Bigg]N \\
& -cF\left(1-\frac{\vec{\pi}^2/2F^2}{1+\vec{\pi}^2/F^2}\right)
\end{aligned}
$$

Adjust constants as dictated by current algebra:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2}\left(1+\frac{\vec{\pi}^2}{F_\pi^2}\right)^{-2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}$$

$$-\bar{N}\Bigg[\gamma^\mu\partial_\mu + m_N$$

$$+ i\gamma^\mu\left(1+\frac{\vec{\pi}^2}{F_\pi^2}\right)^{-1}\Big(2g_A\gamma_5\vec{t}\cdot\partial_\mu\vec{\pi}/F_\pi$$

$$+ 2\vec{t}\cdot(\vec{\pi}\times\partial_\mu\vec{\pi})/F_\pi^2\Big)\Bigg]N$$

$$-m_\pi^2\vec{\pi}^2\left(1+\frac{\vec{\pi}^2}{F_\pi^2}\right)^{-2}$$

Direct approach: SW 1968

For an $SO(4)$ rotation in the $a4$ plane ($a = 1, 2, 3$) by a small angle ϵ, the change in the pion field is

$$\delta\pi_b = -i\epsilon F_\pi \left[\frac{1}{2}\left(1 - \frac{\vec{\pi}^2}{F_\pi^2}\right)\delta_{ab} + \frac{\pi_a \pi_b}{F_\pi^2}\right]$$

and the change in any other field ψ with isospin matrix $\vec{t}$ is

$$\delta\psi = \frac{\epsilon}{F_\pi}\left(\vec{t} \times \vec{\pi}\right)_a \psi$$

Generalization: $G \mapsto H$

Callan, Coleman, Wess, Zumino 1969

Anomalous terms: Wess & Zumino 1971; Witten 1983; D'Hoker & SW 1994

The Standard Model

QCD $\Rightarrow SU(3) \times SU(3) \times U(1)_{\rm B}$, broken only by m_u & m_d & m_s

(with $SU(2) \times SU(2)$ broken only by m_u, m_d)

Electroweak Theory $\Rightarrow W^{\pm}$ & Z^0 & γ couple to $SU(3) \times SU(3) \times U(1)_{\rm B}$ currents.

The success of the Standard Model highlighted the importance of renormalizability. But chiral Lagrangians are non-renormalizable, and so were still considered to be limited to the tree approximation, and justified only by current algebra.

SW 1979:

Non-renormalizable theories are just as renormalizable as renormalizable theories.

In calculating the amplitude for a reaction involving pions and nucleons with kinetic energies of order m_π, a diagram with V_i vertices of type i, L loops, and E_N external nucleon lines, makes a contribution proportional to (kinetic energy)$^\nu$, where

$$\nu = \sum_i V_i \left(\mathcal{I}_i - 2\right) + 2L + 2 - \frac{E_N}{2} \; ,$$

$$\mathcal{I}_i \equiv d_i + \frac{n_i}{2} + m_i$$

and d_i, n_i, and m_i are the numbers of derivatives, nucleon fields, and pion masses at a vertex of type i.

Chiral symmetry $\Rightarrow \mathcal{I}_i \geq 2$.

Leading terms: $L = 0$ and any number of vertices with $\mathcal{I}_i = 2$:

- $n_i = 0$, $d_i = 2$, $m_i = 0$ $\quad (\pi\pi)$
- $n_i = 0$, $d_i = 0$, $m_i = 2$ $\quad (\pi\pi)$
- $n_i = 2$, $d_i = 1$, $m_i = 0$ $\quad (\pi N)$

First Correction: $L = 0$, any number of vertices with $\mathcal{I}_i = 2$, and one vertex with $\mathcal{I}_i = 3$:

- $n_i = 2$, $d_i = 0$, $m_i = 2$ $\quad (\sigma \text{ term})$

Next Correction: Any number of vertices with $\mathcal{I}_i = 2$, and either
1) $L = 0$ and two vertices with $\mathcal{I}_i = 3$, or
2) $L = 1$, or
3) $L = 0$ and one vertex with $\mathcal{I}_i = 4$, acting as counterterm.

But why should we believe these calculations?

Folk Theorem: “if one writes down the most general possible Lagrangian, including *all* terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory , the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition, and the assumed symmetry properties.”

Current algebra is not needed.

Gasser, Leutwyler, Meissner, . . .

Isospin Violation SW 1994, 1996

$$\mathcal{L}_{\rm mass} = V_4 + V_3'$$

$$V_4 = \frac{1}{2}(m_u + m_d)\left(\bar{u}u + \bar{d}d\right)$$

$$V_3' = \frac{1}{2}(m_u - m_d)\left(\bar{u}u - \bar{d}d\right)$$

$m_d/m_u \simeq 1.9$ SW 1977

$$\begin{aligned}\Delta\mathcal{L}_{\rm eff} = & -\frac{A}{2}\left(\frac{1-\pi^2/F_\pi^2}{1+\pi^2/F_\pi^2}\right)\bar{N}N \\ & -B\left[\bar{N}t_3N - \frac{2}{F_\pi^2}\left(\frac{\pi_3}{1+\pi^2/F_\pi^2}\right)\bar{N}\vec{t}\cdot\vec{\pi}N\right] \\ & -\frac{iC}{1+\vec{\pi}^2/F_\pi^2}\bar{N}\gamma_5\vec{\pi}\cdot\vec{t}N \\ & -\frac{iD\pi_3}{1+\vec{\pi}^2/F_\pi^2}\bar{N}\gamma_5 N \qquad (B \simeq -2.5\ \mathrm{MeV})\end{aligned}$$

Nuclear Forces SW 1990, 1991, 1992

Leading terms:

$$\mathcal{I}_i \equiv d_i + \frac{n_i}{2} + m_i = 2 \quad \& \quad L = 0$$

- $n_i = 0$, $d_i = 2$, $m_i = 0$ $\quad (\pi\pi)$
- $n_i = 0$, $d_i = 0$, $m_i = 2$ $\quad (\pi\pi)$
- $n_i = 2$, $d_i = 1$, $m_i = 0$ $\quad (\pi N)$
- $n_i = 4$, $d_i = 0$, $m_i = 0$ $\quad (NN)$

Ordoñez, Van Kolck, Friar, . . .

Effective Field Theories Beyond the Strong Interactions

1. Justifying BCS approximations

Benfatto and Gallavotti, 1990
Feldman and Trubowitz 1990, 1991, 1992
Shankar, 1991, 1993
Polchinski, 1992
SW 1994

2. General Inflation

Cheung, Creminilli, Fitzpatrick, Kaplan,
& Senatore 2008
SW 2009

Is the Standard Model field theory fundamental, or just the leading (renormalizable) term in an effective field theory? Is the underlying theory a quantum field theory?

a. The $SU(3)$, $SU(2)$, and $U(1)$ running couplings (suitably normalized) become almost equal at an energy of order 10^{15} GeV (or 2×10^{16} GeV for SUSY, with improved convergence). This suggests new physics at an energy

$$M \approx 10^{15} \text{ to } 10^{16} \text{ GeV} .$$

b. The Standard Model automatically conserves B and L (without SUSY, or with split SUSY), so there is no need to assume B-conservation and L-conservation are fundamental symmetries. If not fundamental, then we expect interactions:

- $$\frac{g^2}{M}\begin{pmatrix}\nu\\ e\end{pmatrix}\begin{pmatrix}\nu\\ e\end{pmatrix}\begin{pmatrix}\phi^0\\ \phi^+\end{pmatrix}\begin{pmatrix}\phi^0\\ \phi^+\end{pmatrix}$$

  SW 1979

  For $g \approx 1$, $M \approx 10^{16}$ GeV, get $m_\nu \approx 10^{-2}$ eV.

- $$\frac{1}{M^2}\, q\, q\, q\, \ell$$

  SW, Wilczek 1979

Perhaps from GUT, perhaps not.

c. GRAVITATION AS AN EFFECTIVE FIELD THEORY

With an ultraviolet cut-off Λ,

$$I_{\text{eff}} = -\int d^4x \, \sqrt{-\text{Det}g} \Big[f_0(\Lambda) + f_1(\Lambda) R + f_{2a}(\Lambda) R^2 + f_{2b}(\Lambda) R^{\mu\nu} R_{\mu\nu} + f_{3a}(\Lambda) R^3 + \dots \Big]$$

The Λ-dependence of the couplings $f_n(\Lambda)$ is such that physics is independent of Λ.

Large r applications: Donoghue 1994

Introduce dimensionless couplings:

$$g_0 \equiv \Lambda^{-4} f_0\,;\; g_1 \equiv \Lambda^{-2} f_1\,;\; g_{2a} \equiv f_{2a}\,;$$
$$g_{2b} \equiv f_{2b}\,;\; g_{3a} \equiv \Lambda^2 f_{3a}\,;\; \ldots$$

$$\Lambda \frac{d}{d\Lambda} g_n(\Lambda) = \beta_n\Big(g(\Lambda)\Big)$$

Perturbatively most $g_n(\Lambda)$ go to infinity for $\Lambda \to \infty$. They may even become infinite at a *finite* value of Λ.
(Landau 1955; "Triviality" in ϕ^4 theory)
We usually assume that this doesn't matter, because before the couplings blow up, other degrees of freedom (strings? Hořava?) will become important, just as gluons and quarks replace pion and nucleon fields at high energy in chiral dynamics. ‘ But maybe not.

Maybe what you see is what you get.

ASYMPTOTIC SAFETY

The theory is safe from couplings blowing up if $\beta(g_*) = 0$ and $g(\Lambda)$ is on a trajectory attracted to g_*.(SW, 1976)

QCD: $g_* = 0$
More generally, $g_{n*} \neq 0$.

Trajectories with $g \to g_*$ for $\Lambda \to \infty$ form the *ultraviolet critical surface*. The physical requirement that the actual couplings lie on the UV critical surface plays the same role for asymptotically safe theories as does renormalizability*in quantum electrodynamics or quantum chromodynamics.

* no Pauli moment, no $\bar{\psi}\psi\bar{\psi}\psi$

The number of free parameters equals the dimensionality of the UV critical surface. This had better be finite.

For $g \to g_*$,

$$\beta_n(g) \to \sum_m B_{nm}(g_m - g_{*m}) \ , \quad B_{nm} \equiv \left(\frac{\partial \beta_n(g)}{\partial g_m}\right)_*$$

$$g_n(\Lambda) \to g_{n*} + \sum_i u_{in}\, \Lambda^{\lambda_i}$$

where

$$\sum_m B_{nm}\, u_{im} = \lambda_i\, u_{in} \ .$$

The dimensionality of the UV critical surface equals the number of eigenvalues of B_{nm} with negative real part.

Example – 2nd-order phase transition:
$\exists$ fixed point with 1 IR repulsive direction (only need to adjust temperature) so UV critical surface is one-dimensional.

Example – Tricritical point:
$\exists$ fixed point with 2 IR repulsive directions (only need to adjust temperature & pressure) so UV critical surface is two-dimensional.

But in these cases the effective theory breaks down when

$$\Lambda \approx 1/\text{particle separation} .$$

Indications of Asymptotically Safe Gravitation

- Dimensional Continuation ($d = 2 + \epsilon$)
  - SW 1979
  - Kawai, Kitazawa, & Ninomiya, 1993, 1996
  - Aida & Kitazawa, 1997 (2 loops)
  - Niedermaier 2003
- $1/N$ Expansion
  - Smolin 1982 ($R + C^2$)
  - Percacci, 2006
- Lattice Quantization
  - Ambjørn, Jurkewicz, & Loll, 2004, 2005, 2006, 2008

- Truncated ‘Exact” Renormalization Group
  - Wegner & Houghton, 1973
  - Polchinski, 1984
  - Wetterich, 1993

  (Exact renormalization group equations link all $g_n(\Lambda)$. One truncates these equations by setting all but a finite number of $g_n(\Lambda)$ equal zero, ignoring the non-zero value of the $\beta_n(g)$ for which g_n is set equal to zero.)

  - Reuter, 1998
  - Dou & Percacci, 1998 (gravity + free matter)
  - Souma, 1999 ($R + \lambda$, 2 attractive directions)
  - Lauscher & Reuter, 2001 ($R + \lambda$, 2 attractive directions)

– Reuter & Saueressig, 2002 ($R+\lambda$, 2 attractive directions)

– Lauscher & Reuter, 2002 ($R+\lambda+R^2$, 3 attractive directions)

– Reuter & Saueressig, 2002

– Percacci & Perini, 2002, 2003 (constraints on free matter)

– Perini, 2004

– Litim, 2004

– Codello & Percacci, 2006

– Reuter & Saueressig, 2007

– Machado & Saueressig, 2007

– Litim, 2008

With only 2 non-zero couplings, fixed point has 2 UV attractive directions.
With only 3 non-zero couplings, fixed point has 3 UV attractive directions.
This was not encouraging.

Good News!

– Codello, Percacci, & Rahmede, 2008

$$\mathcal{L} = \sqrt{g} \sum_{n=0}^{n_{\max}} f_n R^n$$

For $n_{\max} = 2,3,4,5$, or 6, $\exists$ just 3 attractive directions

– Codello, Percacci, & Rahmede, 2008, same with matter, again 3 attractive directions

– Benedetti, Machado, & Saueressig, 2009, R^0, R, R^2, C^2, 3 attractive directions

– Benedetti, Machado, & Saueressig, 2009, the same with matter, again 3 attractive directions. (No unphysical poles.)

These results suggest the existence of an asymptotically free quantum field theory of gravity with no problems at infinite energy and just three free parameters.

Convergence

Codello, Percacci, & Rahmede, 2008

$$\mathcal{L} = \sqrt{g} \sum_{n=0}^{n_{\rm max}} f_n R^n$$

UV attractive eigenvalues

$n_{\rm max} = 2$	$-1.38 \pm 2.32i$	-26.8
$n_{\rm max} = 3$	$-2.71 \pm 2.27i$	-2.07
$n_{\rm max} = 4$	$-2.86 \pm 2.45i$	-1.55
$n_{\rm max} = 5$	$-2.53 \pm 2.69i$	-1.78
$n_{\rm max} = 6$	$-2.41 \pm 2.42i$	-1.50

In terms of dimensionless couplings,

$$I_{\rm eff} = -\int d^4x\,\sqrt{-{\rm Det}g}\Big[\Lambda^4 g_0(\Lambda) + \Lambda^2 g_1(\Lambda) R$$
$$+g_{2a}(\Lambda)R^2 + g_{2b}(\Lambda)R^{\mu\nu}R_{\mu\nu}$$
$$+\Lambda^{-2}g_{3a}(\Lambda)R^3 + \ldots\Big]$$

Physics is independent of Λ, but only if we include radiative corrections (loop graphs) with an ultraviolet cutoff Λ. This is hard. But we can choose Λ to minimize radiative corrections, and use $I_{\rm eff}$ in tree approximation to derive the field equations.

For S-matrix elements, radiative corrections are small if Λ is less than typical external momenta. For vacuum amplitudes in presence of background Robertson–Walker metric, it is plausible that radiative corrections are small if $\Lambda < H$. ($H \equiv \dot{a}/a$.) If H is sufficiently large, we can also take Λ large enough so that

$$g_n(\Lambda) \simeq g_{n*} \; .$$

There is always a de Sitter solution:

$$d\tau^2 = dt^2 - e^{2Ht} d\vec{x}^{\,2} \; .$$

$$2g_{0*} + \left(\frac{R}{\Lambda^2}\right) g_{1*} - \left(\frac{R}{\Lambda^2}\right)^3 g_{3a*} + \ldots = 0 \; ,$$

$$R = -12H^2 \; .$$

We need a root with $|R|/\Lambda^2 > 12$.

For this solution, inflation never ends.

But there are also solutions with a more general Robertson–Walker metric:

$$d\tau^2 = dt^2 - a^2(t)d\vec{x}^{\,2}$$

for which H decreases with time, and

$$2g_{0*}+\left(\frac{R}{\Lambda^2}\right)g_{1*}-\left(\frac{R}{\Lambda^2}\right)^3 g_{3a*}+\ldots\neq 0\ ,$$

$$R = -12H^2 - 6\dot{H}$$

Example:

$$I_{\rm eff}[g,\Lambda] = -\Lambda^4\int d^4x\,\sqrt{-{\rm Det}g}\,F(R)$$

$$\begin{aligned} F(R) \equiv\ & g_{0*} + (R/\Lambda^2)g_{1*} + (R/\Lambda^2)^2 g_{2*} \\ & +(R/\Lambda^2)^3 g_{3*} + \ldots \end{aligned}$$

General field equation:

$$F(R)R_{\mu\nu} + F'(R)R_{,\mu;\nu} + F'''(R)R_{,\mu}R_{,\nu}$$
$$= g_{\mu\nu}\left[\frac{1}{2}F(R) + F''(R)R_{,\lambda;\sigma}\, g^{\lambda\sigma}\right.$$
$$\left. + F'''(R)R_{,\lambda}R_{,\rho}\, g^{\lambda\rho}\right] .$$

For a Robertson-Walker solution, we only need 00 component of field equation:

$$0 = H\dot{R}F''(R) - (\dot{H} + H^2)F'(R) - \frac{1}{6}F(R)$$

$$R = -12H^2 - 6\dot{H}$$

Condition for de Sitter solution:

$$RF'(R)/F(R) = 2\;, \quad R = -12H^2$$

If $RF'(R)/F(R)$ is only near 2, $\exists$ a solution with H nearly constant, and

$$\frac{\dot{H}}{H^2} \simeq -\frac{1}{3}\left(\frac{RF'(R)}{F(R)} - 2\right)$$

H decreases slowly if $RF'(R)/F(R)$ is a little greater than 2. Eventually H would fall below Λ, and radiative corrections become important. But we can reduce Λ to restore the ratio of H/Λ, and continue the slow decrease of H.*

* In a different model, Bonanno & Reuter (2002, 2004) and Reuter & Saueressig (2005) change Λ continuosly.

As long as H changes slowly, and $g_n(\Lambda) \simeq g_{n*}$, the quantity $RF'(R)/F(R)$ depends only on H/Λ, so this quantity stays near 2 as we reduce Λ to keep H/Λ constant. But eventually Λ becomes small enough so that $g_n(\Lambda)$ moves away from g_{n*}, and then reducing Λ further to keep $\Lambda < H$ will change $RF'(R)/F(R)$, and so inflation ends.