Bayesian parameter estimation in effective field theories

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Low-energy constants (LECs) in effective field theory

Generic physical quantity

$$A = A^{(0)} + A^{(1)}\frac{m}{\Lambda} + A^{(2)}\left(\frac{m}{\Lambda}\right)^2 + \cdots$$

LECs $A^{(i)}$

- Encode effects of high-energy physics
- Need to be known to make model-independent predictions
- Requirement: LECs are "natural" or "$\mathcal{O}(1)$"
- Determined from
  - Underlying theory $\rightarrow$ not always possible
  - Data

Bayes' theorem in data analysis

$$\mathrm{pr}(A|B) = \frac{\mathrm{pr}(B|A)\mathrm{pr}(A)}{\mathrm{pr}(B)}$$

$$\underbrace{\mathrm{pr}(\mathit{theory}|\mathit{data})}_{\mathit{posterior}} \propto \underbrace{\mathrm{pr}(\mathit{data}|\mathit{theory})}_{\mathit{likelihood}} \times \underbrace{\mathrm{pr}(\mathit{theory})}_{\mathit{prior}}$$

- Posterior: Given the data, what is the information to be gained on the theory (parameter values)?
- Likelihood: Probability of observing the data given the theory is true (particular parameter set)
  $\rightarrow$ Only part considered in "standard" statistics
- Prior: Incorporates any knowledge about the theory before analysis of the data
  Example: Measurement of mass $m > 0 \rightarrow \mathrm{pr}(m) = \Theta(m)$

Marginalization

- Theory can contain a large number of parameters
- Not necessarily interested in all parameters (e.g. higher-order constants, background signal,...)
- Can "integrate out" these nuisance parameters: Marginalization

$$\mathrm{pr}(A, B|C) \mapsto \mathrm{pr}(A|C) = \int dB\, \mathrm{pr}(A, B|C)$$
$$= \int dB\, \mathrm{pr}(A|B, C) \times \mathrm{pr}(B|C)$$

Data analysis for EFTs

Which order in the m/Λ expansion is appropriate?

- There is no “right” order in the expansion
  $\Rightarrow$ Marginalize over order
  $\Rightarrow$ Uncertainty included in parameter estimates

How does one incorporate information on naturalness of LECs?

- Use Bayes’ theorem to include naturalness via a prior pdf
  E.g. $\mathrm{pr}(\mathbf{a}|M,R) = \left(\frac{1}{\sqrt{2\pi}R}\right)^{M+1} \exp\left(-\frac{\mathbf{a}^2}{2R^2}\right)$
- Notion of $\mathcal{O}(1)$ not precise
  $\Rightarrow$ Marginalize over R to take uncertainty into account

Higher-order parameters

- Not interested in actual values
  $\Rightarrow$ Marginalize over higher-order LECs
  $\Rightarrow$ Systematic treatment of related uncertainty

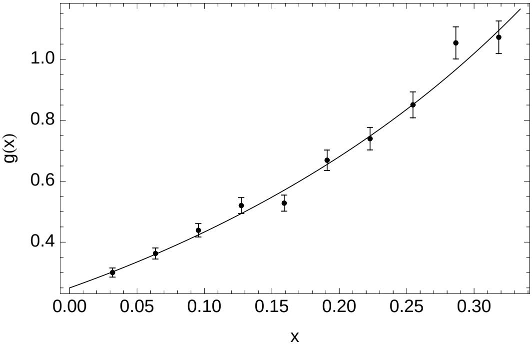
Final probability density function

$$\mathrm{pr}(\mathbf{a}_{res}|D) = \sum_{P=P_{min}}^{P_{max}} \int_{R_{min}}^{R_{max}} dR \int d\mathbf{a}_{marg}\, \frac{\mathrm{pr}(D|\mathbf{a},P)\mathrm{pr}(\mathbf{a}|P,R)\mathrm{pr}(P)\mathrm{pr}(R)}{\mathrm{pr}(D)}$$

- $\mathrm{pr}(\mathbf{a}_{res}|D)$: posterior pdf for parameter subset $\mathbf{a}_{res}$
- $\mathrm{pr}(D|\mathbf{a},P)$: likelihood, considered in "standard" approach
- $\mathrm{pr}(\mathbf{a}|P,R)$: prior pdf, incorporates naturalness
- $\sum_P \mathrm{pr}(P)$: marginalization over EFT order P with prior
- $\int dR\, \mathrm{pr}(R)$: marginalization over "naturalness parameter" R with prior
- $\int d\mathbf{a}_{marg}$: marginalization over higher-order parameters

Parameter estimation: Toy model

- Some data are given:



- Goal: Fit polynomial $f(x) = \sum_{i=0}^{P} a_i x^i$ to data and determine first two coefficients a_0, a_1
- Information:
  - Data is Gaussian distributed
  - Coefficients are "natural", i. e. $\mathcal{O}(1)$

Results

Standard least-squares approach:

P	$\chi^2/d.o.f.$	a_0	a_1
1	2.23	0.203 ± 0.014	2.51 ± 0.10
2	1.06	0.260 ± 0.022	1.31 ± 0.39
3	1.13	0.235 ± 0.038	2.14 ± 1.08
4	1.13	0.177 ± 0.067	4.76 ± 2.70
5	0.99	0.327 ± 0.133	-3.56 ± 6.94
6	1.32	0.314 ± 0.297	-2.73 ± 18.5
7	1.47	1.05 ± 0.792	-56.3 ± 56.5

Bayesian approach ($P = 2 - 8$, $R = 0.1 - 10$):

$$a_0 = 0.246 \pm 0.021, \quad a_1 = 1.63 \pm 0.37$$

"True" values:

- Underlying function $F(x) = \left[\frac{1}{2} + \tan\left(\frac{\pi}{2}x\right)\right]^2$
- Low-x expansion: $a_0 = 0.25,\ a_1 \approx 1.57$

Lattice data: Nucleon mass

$$M_{\chi PT}(m) = M_0 + k_1 m^2 + k_2 m^3 + k_3 m^4 \log\left(\frac{m}{\mu}\right) + k_4 m^4 \ldots$$

- # paramaters $M >$ chiral order P
- For ‘naturalness prior’ larger values of R are suppressed
- Numerical values of dimensionless couplings depends on underlying scale Λ
  $\Rightarrow$ Sensitivity on choice of Λ
- More detailed information on k_i available
  - Pion decay constant f
  - Axial coupling g_A
  - Mesonic LECs l_i
  - Nucleonic LECs c_i

J. A. McGovern, D. R. Phillips, M.R.S., in preparation

Summary & Outlook

- Parameter estimation from data → important component of effective field theory programs
- Bayes' theorem and marginalization → powerful alternative to standard approach
  - Incorporate prior information (e.g. naturalness)
  - Eliminate nuisance parameters
  - Avoid (arbitrary) choice of order of EFT calculation
  - Systematic treatment of uncertainties

- Apply to real data
- Generalize to non-linear dependence on parameters
- Theories with multiple scales
- Determination of higher-order parameters?
  - Refine priors
  - How well do low-order parameters have to be known?