Electromagnetic corrections in $\eta \to 3\pi$ decays

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Why are the decays $\eta \longrightarrow 3\pi$ interesting?

- $\eta \longrightarrow 3\pi$ forbidden by **isospin symmetry**, two sources of breaking:

$$\mathcal{H}_{\text{QCD}}(x) = \frac{m_d - m_u}{2}(\bar{d}d - \bar{u}u)(x)$$
$$\mathcal{H}_{\text{QED}}(x) = -\frac{e^2}{2}\int dy\, D^{\mu\nu}(x-y)T(j_\mu(x)j_\nu(y))$$

- electromagnetic effects are small: **approx.** $\mathcal{A}_{\eta\to 3\pi} \sim (m_d - m_u)$ [Sutherland 1966]

  $\Rightarrow$ **clean access** to quark mass ratios

- systematic machinery must cope with both effects accurately

  $\Rightarrow$ **chiral perturbation theory with virtual photons**

- many **testable predictions**: Dalitz plot parameters, branching ratio $\frac{\Gamma(\eta\to 3\pi^0)}{\Gamma(\eta\to\pi^0\pi^+\pi^-)}$

- new high statistics experiments: [WASA@COSY, CB@MAMI-B/-C, KLOE/KLOE-2@DAΦNE]

  $\Rightarrow$ reconsider electromagnetic corrections to achieve higher precision

What do we know by now?

$\Gamma(\eta \longrightarrow 3\pi)$: calculations vs. experiments

- NLO strong effects $\mathcal{O}((m_d - m_u)p^2)$ a factor two off from exp. [Gasser/Leutwyler (GL) 1985]
  $\Rightarrow$ large unitary corrections (FSI: $\pi\pi$ **rescattering**)
- strong unitary corrections beyond $\mathcal{O}(p^4)$ obtained via
  dispersive techniques [Anisovich/Leutwyler & Kambor et.al. 1996]
  and UChPT [Borasoy/Nißler 2005]
- NNLO strong effects $\mathcal{O}((m_d - m_u)p^4)$ enhance NLO result [Bijnens/Ghorbani 2007]

- LO em contributions $\mathcal{O}(e^2)$ vanish [Sutherland1966]
- NLO em effects $\mathcal{O}(e^2p^2)$ found to be small [Bauer/Kambor/Wyler (BKW) 1996]

So why should one reconsider the electromagnetic corrections?

- BKW neglected $\mathcal{O}(e^2(m_d - m_u))$ as it is of **2^{nd} order** in isospin breaking, but:

  $\Rightarrow$ neither **photon loops** nor **pion mass difference** [$\hookrightarrow$ approx. $M^2_{\pi^\pm} - M^2_{\pi^0} \sim e^2$]

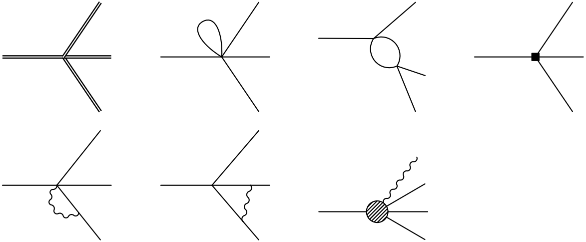
  $\Rightarrow$ hence no non-trivial analytic structures in amplitudes

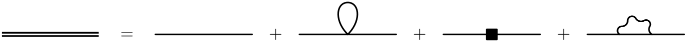
- expected features at considered order:
  - **Coulomb pole** at threshold $s = 4M^2_{\pi^\pm}$ in charged amplitude
  - **cusps** at $s, t, u = 4M^2_{\pi^\pm}$ in neutral amplitude due to $\eta \to \pi^0 \boldsymbol{\pi^+ \pi^-} \to \pi^0 \boldsymbol{\pi^0 \pi^0}$

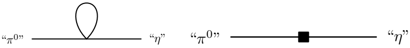
- DKM corrections should be roughly $\frac{m_d - m_u}{m_d + m_u} \approx 1/3$ compared with BKW corrections

$\eta \longrightarrow 3\pi$ up to NLO at $\mathcal{O}(e^2(m_d - m_u))$

- strong and electromagnetic diagrams:



- full propagator: 
- LO $\pi^0\eta$ mixing: $\varepsilon \sim (m_d - m_u)$

NLO $\pi^0\eta$ mixing: 

- LECs: some strong L_i (known), some em K_i (dimensional analysis)

Results: General Remarks

- $\Delta I = 1$ relation **not valid** at **2nd order** in isospin breaking:

  $\mathcal{A}_n(s,t,u) \neq \mathcal{A}_c(s,t,u) + \mathcal{A}_c(t,u,s) + \mathcal{A}_c(u,s,t)$ [$\hookrightarrow$ e.g. photon loops]

- all results **relative** to GL amplitude $\mathcal{O}((m_d - m_u)p^2)$ [GL, Nucl. Phys. B **250** (1985) 539]

- no imaginary effects from BKW amplitude $\mathcal{O}(e^2 p^2)$ [BKW, Nucl. Phys. B **460** (1996) 127]

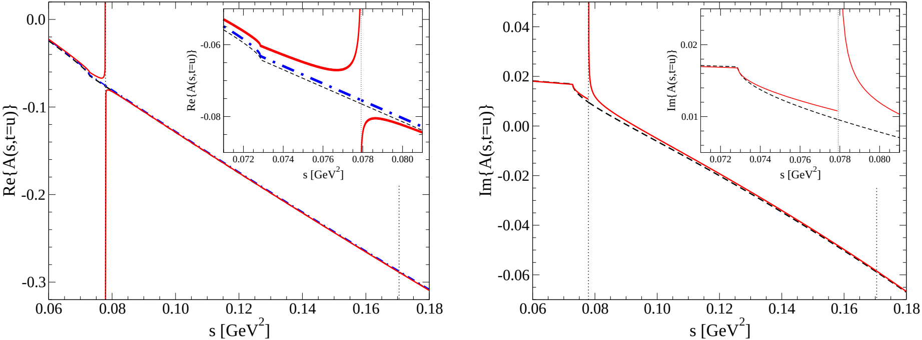
- **error estimates** from variation of em LECs K_i [$\hookrightarrow$ not from higher orders in ChPT]

- IR & kinematical divergences: soft-photon bremsstrahlung

  $\Rightarrow$ subtraction of **universal soft-photon corrections**

$\eta \longrightarrow \pi^0\pi^+\pi^-$ Decay Amplitude

real and **imaginary** part of NLO charged decay **amplitude** along $t = u$ line:

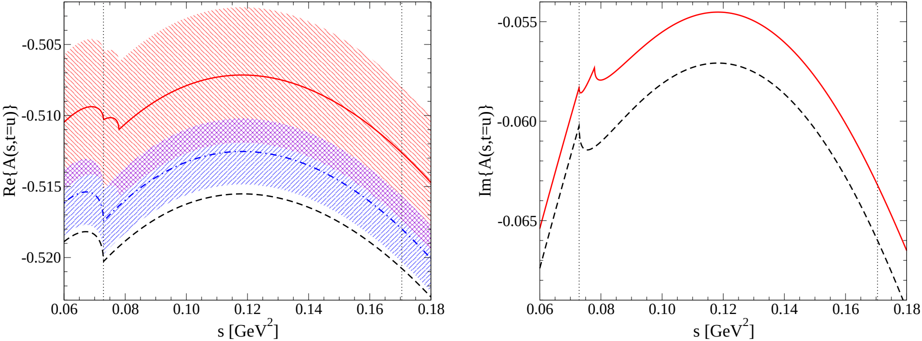


GL, GL+BKW, GL+BKW+DKM

- uncertainties hardly visible since $\mathcal{A}^{LO}_{\eta\to\pi^0\pi^+\pi^-} \sim -s$ [$\hookrightarrow$ line widths]
- illustration of divergences: IR cured by hand, **Coulomb pole & phase** retained

$\eta \longrightarrow 3\pi^0$ Decay Amplitude

real and **imaginary** part of NLO neutral decay **amplitude** along $t = u$ line:



GL, GL+BKW, GL+BKW+DKM

- notice **small scale** due to $\mathcal{A}^{LO}_{\eta\to 3\pi^0} \sim$ **const.** [$\hookrightarrow$ visible error bands]
- **cusp** at $\pi^+\pi^-$ threshold clearly visible
- **sizes** of DKM and BKW corrections **comparable**

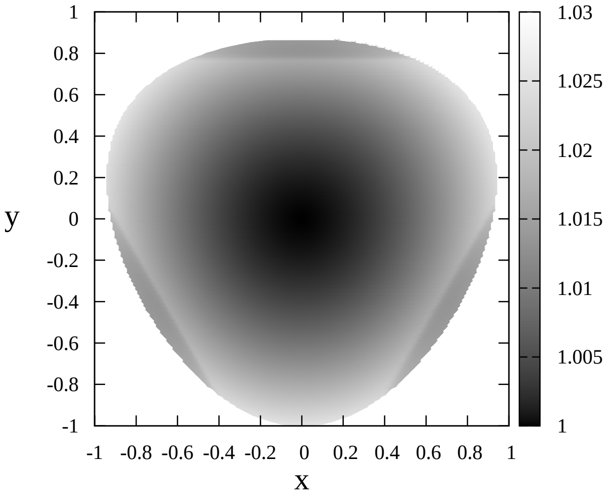
$$|\mathcal{A}_c(x,y)|^2 = |\mathcal{N}_c|^2\{1 + ay + by^2 + dx^2 + fy^3 + gx^2y + \dots\}$$

$$x = \frac{\sqrt{3}(u-t)}{2M_\eta Q_c} \qquad y = \frac{3\left[(M_\eta - M_{\pi^0})^2 - s\right]}{2M_\eta Q_c} - 1$$

	$\lvert\mathcal{N}_c\rvert^2$	a	b	d
GL	0.0325	−1.279	0.396	0.0744
ΔBKW	$(-1.1 \pm 0.9)\%$	$(+0.6 \pm 0.1)\%$	$(+1.4 \pm 0.2)\%$	$(+1.5 \pm 0.5)\%$
ΔDKM	$(-2.4 \pm 0.7^*)\%$	$(+0.7 \pm 0.4)\%$	$(+1.5 \pm 0.7)\%$	$(+4.4 \pm 0.4^*)\%$

- all corrections at percent level: normalization reduced, slopes increased
- DKM: universal soft-photon corrections subtracted

Cusps in $\eta \longrightarrow 3\pi^0$ Dalitz Plot



- **cusps** at $\pi^+\pi^-$ threshold in s, t and u clearly visible

Dalitz Plot Parameters for $\eta \longrightarrow 3\pi^0$

$$|\mathcal{A}_n(x,y)|^2 = |\mathcal{N}_n|^2 \{1 + 2\alpha z + \dots\}$$

$$z = x^2 + y^2$$

	$\lvert\mathcal{N}_n\rvert^2$	$10^2 \times \alpha$	χ^2 / ndf
GL	0.269	1.27	$\equiv 1$
ΔBKW	$(-1.1 \pm 0.9)\%$	$(+3.7 \pm 0.5)\%$	0.99
ΔDKM	$(\mathbf{-3.3} \pm 1.8)\%$	$(\mathbf{-0.2} \pm 1.0)\%$	$\mathbf{6.20}$
ΔDKM**(cusp)**	$(-3.3 \pm 1.8)\%$	$(\mathbf{+5.0} \pm 1.1)\%$	0.35

- simple polynomial fit in z can not account for **cusp** structures [$\hookrightarrow$ c.f. χ^2 / ndf]

  $\Rightarrow$ α gets reduced by roughly 4% [$\hookrightarrow$ but effect too small to explain sign-discrepancy]

- DKM**(cusp)**: fit of inner region $z \leq z_{\text{cusp}}$ excluding cusps

Summary & Outlook

- electromagnetic corrections in general small (but need to be accounted for),

  DKM effects at $\mathcal{O}(e^2(m_d - m_u))$ as large as BKW effects at $\mathcal{O}(e^2\hat{m})$
- observe non-trivial analytic structure with **Coulomb pole** and **cusps**
- calculated **new corrections** for many observables:

  Dalitz plot parameters, *decay widths, branching ratio, quark mass ratios*
- timely for new high statistics experiments:
  [WASA@COSY, Phys. Lett. B **677** (2009) 24] *Kupść*
  [CB@MAMI-B, Eur. Phys. J. A **39** (2009) 169]
  [CB@MAMI-C, Phys. Rev. C **79** (2009) 035204] *Prakhov*
  [KLOE/KLOE-2@DAΦNE] *Jacewicz*
- theoretical framework perfectly suited for extraction of $\pi\pi$ scattering lenghts is

  non-relativistic effective field theory:
  [Bissegger *et al.*, Phys. Lett. B **659** (2008) 576]
  [Bissegger *et al.*, Nucl. Phys. B **806** (2009) 178]
  [Gullström/Kupść/Rusetsky, Phys. Rev. C **79** (2009) 028201]
- new dispersive analysis of $\eta \to 3\pi^0$: *Lanz*

Chiral Perturbation Theory with Virtual Photons

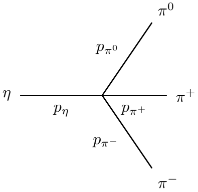
[Weinberg 1979, Gasser/Leutwyler 1984/1985, Urech 1995, ...]

$$\mathcal{L}_{\text{eff}} = \sum_{n=1}^{\infty} \mathcal{L}^{(2n)} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots$$

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4} \langle D^\mu U^\dagger D_\mu U + \chi^\dagger U + U^\dagger \chi \rangle + C \langle \mathcal{Q} U \mathcal{Q} U^\dagger \rangle - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

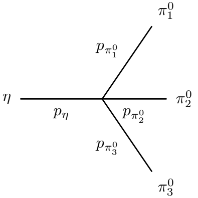
- $U = \exp\left(\frac{i\phi}{F_0}\right) \qquad \chi \sim 2B_0 \, \text{diag}(m_u, m_d, m_s) \qquad \mathcal{Q} = \frac{e}{3} \, \text{diag}(2, -1, -1)$
  $D_\mu \sim \partial_\mu - eQA_\mu \qquad F_0 \sim F_\pi \qquad B_0 \sim |\langle 0|\bar{q}q|0\rangle|/F_0^2$
- C fixed from pion mass difference $\Delta M_\pi^2 = (M_{\pi^\pm}^2 - M_{\pi^0}^2)^{\text{LO}} = (2e^2 C)/F_0^2$
- for $m_u = m_d \Rightarrow \Delta M_K^2 = \Delta M_\pi^2$ [Dashen 1969]
- for $m_u \neq m_d \Rightarrow (\Delta M_K^2)_{\text{str}} = -(m_d - m_u) B_0$
- $\eta\pi^0$ mixing described at LO by angle $\epsilon = \frac{1}{2} \arctan\left(\frac{\sqrt{3}}{2} \frac{m_d - m_u}{m_s - \hat{m}}\right) \qquad \hat{m} = \frac{m_d + m_u}{2}$,
  $\eta\eta'$ mixing encoded in strong NLO LEC L_7

$$\mathcal{A}^{LO}_{\eta\to\pi^0\pi^+\pi^-} = -\frac{B_0(m_d - m_u)}{3\sqrt{3}F_\pi^2}\left\{1 + \frac{3(s - s_0^c) + 2\Delta M_\pi^2}{M_\eta^2 - M_{\pi^0}^2}\right\}$$



$$s = (p_\eta - p_{\pi^0})^2\,,\quad t = (p_\eta - p_{\pi^+})^2\,,\quad u = (p_\eta - p_{\pi^-})^2$$

$$\mathcal{A}^{LO}_{\eta\to 3\pi^0} = -\frac{B_0(m_d - m_u)}{3\sqrt{3}F_\pi^2}$$



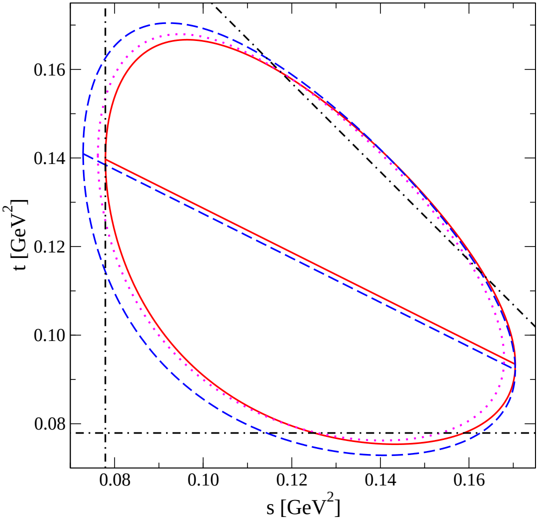
$$s = (p_\eta - p_{\pi^0_1})^2\,,\quad t = (p_\eta - p_{\pi^0_2})^2\,,\quad u = (p_\eta - p_{\pi^0_3})^2$$

$$3s_0^{n[c]} = s+t+u = M_\eta^2 + 3M_{\pi^0}^2\ [+2\Delta M_\pi^2]$$

$$\Rightarrow\quad \Gamma(\eta\to 3\pi)^{\mathrm{LO}} \sim Q^{-4}\,,\qquad Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$$

- notice: $\mathcal{A}^{LO}_{\eta\to\pi^0\pi^+\pi^-} \sim -s\,,\quad \mathcal{A}^{LO}_{\eta\to 3\pi^0} \sim$ **const.**

Kinematical Bounds of Dalitz Plots



- $\eta \longrightarrow \pi^0\pi^+\pi^-$
- $\eta \longrightarrow 3\pi^0$
- $\eta \longrightarrow 3\pi$ with $\overline{M_\pi^2} = \frac{(M_{\pi^0}^2 + 2M_{\pi^\pm}^2)}{3}$
- $4M_{\pi^\pm}^2$ **cusp lines** and **$t = u$ lines**

$\Rightarrow$ sizeable deviations at edges of Dalitz plots

$\eta \to \pi^0\pi^+\pi^-$ Dalitz Slopes and Normalization

$$|\mathcal{A}_c(x,y)|^2 = |\mathcal{N}_c|^2\{1 + ay + by^2 + dx^2 + fy^3 + gx^2y + \ldots\}$$

	$\lvert\mathcal{N}_c\rvert^2$	a	b	
GL	0.0325	-1.279	0.396	
ΔBKW	-0.0004 ± 0.0003	-0.008 ± 0.001	$+0.006 \pm 0.001$	
	$= (-1.1 \pm 0.9)\%$	$= (+0.6 \pm 0.1)\%$	$= (+1.4 \pm 0.2)\%$	
ΔDKM	$-0.0008 \pm 0.0002^*$	-0.009 ± 0.005	$+0.006 \pm 0.003$	
	$= (-2.4 \pm 0.7^*)\%$	$= (+0.7 \pm 0.4)\%$	$= (+1.5 \pm 0.7)\%$	
	d	f	g	χ^2 / ndf
GL	0.0744	0.0126	-0.0586	$\equiv 1$
ΔBKW	$+0.0011 \pm 0.0004$	-0.0003 ± 0.0001	-0.0010 ± 0.0003	1.03
	$= (+1.5 \pm 0.5)\%$	$= (-2.2 \pm 0.4)\%$	$= (+1.7 \pm 0.6)\%$	
ΔDKM	$+0.0033 \pm 0.0003^*$	$+0.0001 \pm 0.0001$	$-0.0038 \pm 0.0009^*$	1.63
	$= (+4.4 \pm 0.4^*)\%$	$= (+0.5 \pm 0.6)\%$	$= (+6.4 \pm 1.5^*)\%$	

$\eta \longrightarrow 3\pi^0$ Dalitz Slope and Normalization

$$|\mathcal{A}_n(x,y)|^2 = |\mathcal{N}_n|^2 \{1 + 2\alpha z + \ldots\}$$

	$\lvert\mathcal{N}_n\rvert^2$	$10^2 \times \alpha$	$\chi^2 /$ ndf
GL	0.269	1.27	$\equiv 1$
ΔBKW	-0.003 ± 0.002	$+0.05 \pm 0.01$	0.99
	$= (-1.1 \pm 0.9)\%$	$= (+3.7 \pm 0.5)\%$	
ΔDKM	-0.009 ± 0.005	-0.002 ± 0.01	**6.20**
	$= (\mathbf{-3.3} \pm 1.8)\%$	$= (\mathbf{-0.2} \pm 1.0)\%$	
ΔDKM**(cusp)**	-0.009 ± 0.005	$+0.06 \pm 0.01$	0.35
	$= (-3.3 \pm 1.8)\%$	$= (\mathbf{+5.0} \pm 1.1)\%$	

Decay Widths, Branching Ratio and Quark Mass Ratio Q

	$\eta \to \pi^0\pi^+\pi^-$	$\eta \to 3\pi^0$
Γ^{GL}	$154.5\,\text{eV}$	$222.8\,\text{eV}$
$\Delta\Gamma^{\text{BKW}}$	$(-1.0 \pm 0.9)\%$	$(-1.1 \pm 0.9)\%$
$\Delta\Gamma^{\text{DKM}}$	$(-1.9 \pm 0.5^*)\%$	$(\mathbf{-3.3} \pm 1.8)\%$
$\Delta\Gamma^{\text{DKM(uc)}}$	$(-1.0 \pm 0.5^*)\%$	

r^{GL}	1.442
Δr^{BKW}	$(-0.1 \pm 1.2)\%$
Δr^{DKM}	$(\mathbf{-1.4 \pm 1.8})\%$
$\Delta r^{\text{DKM(uc)}}$	$(-2.3 \pm 1.8)\%$

	$\eta \to \pi^0\pi^+\pi^-$	$\eta \to 3\pi^0$
ΔQ^{BKW}	$(+0.24 \pm 0.22)\%$	$(+0.28 \pm 0.22)\%$
ΔQ^{DKM}	$(+0.48 \pm 0.12^*)\%$	$(\mathbf{+0.84} \pm 0.46)\%$
$\Delta Q^{\text{DKM(uc)}}$	$(+0.24 \pm 0.12^*)\%$	

- DKM**(uc)**: **no** subtraction of universal soft-photon corrections
- branching ratio: $r = \frac{\Gamma(\eta \to 3\pi^0)}{\Gamma(\eta \to \pi^0\pi^+\pi^-)}$
- use approx. $\Gamma \sim Q^{-4}$

[$\hookrightarrow$ does not hold for BKW terms $\mathcal{O}(e^2\hat{m})$]

[$\hookrightarrow$ input value $Q^{\text{GL}} = Q^{\text{Dashen}} = 24.2$]

$\Rightarrow$ apply opposite shift to purify extraction of Q from experiment